

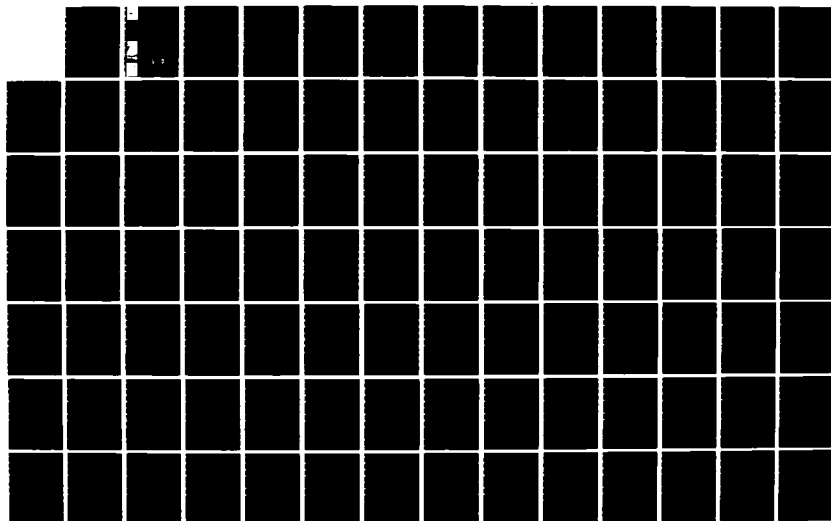
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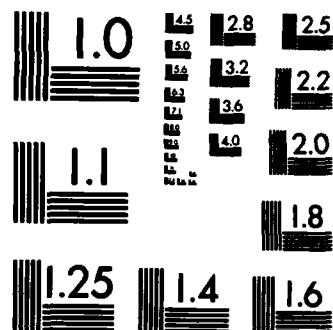
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INVESTIGATION OF TIDALLY INDUCED TURBULENT FLOW

by

Norman W. Scheffner

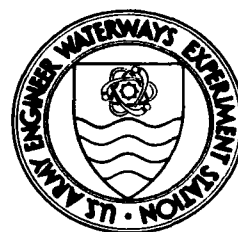
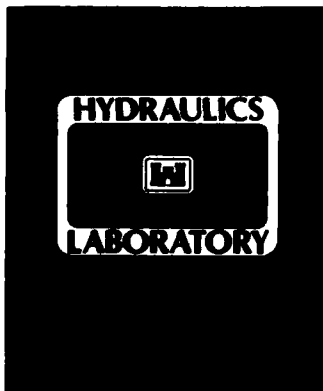
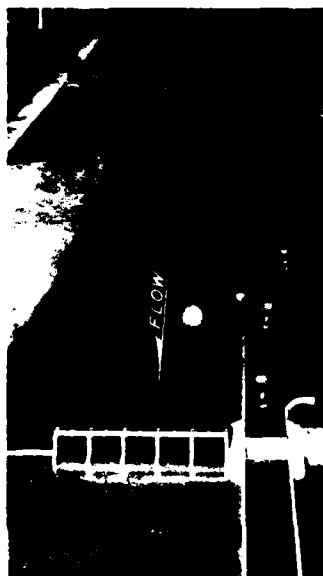
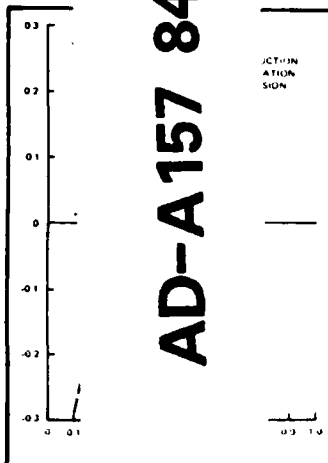
Hydraulics Laboratory

DEPARTMENT OF THE ARMY
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PO Box 631
Vicksburg, Mississippi 39180-0631



US Army Corps
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March 1985

Final Report

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Under Project 4A161101A91D

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PREFACE

The present investigation was funded through the In-House Laboratory Independent Research (ILIR) program offered at the US Army Engineer Waterways Experiment Station (WES). The work unit, funded as Project Number 4A161101A91D, was originally entitled "Investigation of Tidal Constituent Phenomena in Density Stratified Flow." This original scope of work was narrowed so that a thorough analysis of the basic mechanics of tidally induced stratified flow could be realized. This was performed by an investigation of tidally induced turbulent flow.

This project was conducted during the period of 1 January 1981 through 30 September 1983. The study was conducted by Mr. Norman W. Scheffner of the Estuaries Division, under the direction of Mr. R. A. Sager, Chief of the Estuaries Division, Hydraulics Laboratory, under the direction of Mr. H. P. Simmons, former Chief of the Hydraulics Laboratory of WES. Dr. R. H. Multer provided valuable technical guidance during the formulation of the project. Mr. E. C. McNair, former Chief of the Sedimentation Branch of the Estuaries Division, provided support during the execution of the project and the preparation of the report. The final report was written by Mr. Scheffner.

Commanders and Directors of WES during this investigation and the preparation and publication of this report were COL Nelson P. Conover, CE, COL Tilford C. Creel, CE, and COL Robert C. Lee, CE. Technical Director was Mr. F. R. Brown

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SECTION 1

INTRODUCTION

The hydrodynamics of estuaries and tidal inlets are extremely complicated, often requiring both numerical and physical model simulations to address certain problems or potential problems. One of the most difficult of these problems is that associated with density stratification in which fresh water overrides the denser, more saline seawater. The original (December 1980) In-House Laboratory Independent Research (ILIR) proposal was aimed at addressing this problem of density stratified flow. A thorough analysis of the interrelated phenomena associated with this type of flow regime showed that a thorough analysis of the basic turbulent flow regime associated with tidally induced flow systems was first necessary before any significant progress in density stratification could be made. For example, the flow-induced turbulence in a particular flow regime can result in mixing of the fresh and saline water, producing a well-mixed condition. In this case, density stratification does not exist. The existence, partial existence, or nonexistence of density stratification is therefore directly related to the basic parameters of the flow regime and to how these flow parameters

affect the level of turbulence in the flow system. For these reasons, the original scope of work was narrowed to an analysis of turbulent flow for a tidally induced flow system. The present analysis does not include the effects of density stratification since a complete understanding of the hydrodynamics of turbulent, unstratified, tidally induced flow is first necessary before any meaningful analysis of density effects can be made. Whether or not density stratification is considered, however, the equations governing turbulent flow must still be numerically approximated. If stratified flow is then to be considered, separate equations must be provided to solve the convection/diffusion equation for salinity and an equation of state must be provided to relate salinity to density. Therefore it can be seen that the present analysis of turbulent flow is not inapplicable to stratified flow but merely provides the first, and most difficult, step necessary for analyzing it.

Hydrodynamic numerical models approximating tidally induced turbulent flow regimes are commonly utilized for the evaluation of the future effects of some alteration to the existing flow system. Essentially all large hydrodynamic codes presently in use approximate the turbulent fluctuation portion of the Reynolds equations (Turbulent Navier-Stokes equations) through the use of empirical relationships. Common examples are those of the eddy viscosity or "effective" viscosity type in which the turbulent velocity fluctuations

(Reynolds stresses) are assumed proportional to the mean velocity gradient in a manner analogous to the viscous stresses. This is done through the use of an "effective" viscosity coefficient which accounts for both the viscous stresses and the turbulent stresses. These relationships do not realistically account for the turbulent flow but instead employ certain geometry and flow-dependent coefficient values such that the numerical model results match the observed prototype behavior. Thorough descriptions of models of this type are presented by Launder and Spaulding (1972) and Rodi (1980). This approach is acceptable for many of the simpler flow regimes in which turbulence-driven secondary flows are not of primary interest. If, however, the flow regime is a complex one (such as the case of an estuary) or the proposed changes to be investigated are substantially different from the existing conditions, it can be seen that this empirical approach will not adequately represent the turbulent flow system. The primary reason for this failure is that the "effective" viscosity models do not account for the transport of turbulence through convection or diffusion. When these mechanisms are not accounted for, turbulence-driven secondary flows cannot be predicted to an adequate degree. If advances are to be made in the field of numerical modeling of estuarine systems, models which are capable of reproducing the structure of the turbulent quantities in addition to the mean flow quantities will have to be developed and utilized.

There are presently three approaches to the development of turbulence modeling which are potentially applicable to complex flow regimes. A brief description of these models is presented below.

1. The k- ϵ Model. This approach to the modeling of turbulence represents a very advanced effective viscosity model in which the effective viscosity is represented in terms of turbulent quantities. These quantities, the turbulent kinetic energy k and the isotropic dissipation ϵ , are determined from separate transport equations. A thorough description of this model in addition to various applications of it can be found in Rodi (1980). An additional example of an application of this model to a long wave in an open channel is presented by Smith and Takhar (1979). This particular model represents a significant advance in earlier turbulent models since the effective viscosity is based on temporally and spatially varying turbulent quantities; however, it does not deal with the actual governing equations of turbulent flow. For this reason, the applicability may be limited to only the more basic flow systems.
2. The Reynolds Stress Transport Model. This approach attempts to solve, or approximate, the exact equations derived for the transport of the individual Reynolds stresses. It represents the most rigorous approach that can be taken and is presently the state-of-the-art

numerical treatment of turbulent flow. Because of the additional equations and variables involved, it also represents the most expensive approach. For this reason, the Reynolds stress transport models have been applied only to steady-state flow conditions. Excellent examples of this formulation have been presented by Hanjalic and Launder (1972), Launder, Reece, and Rodi (1975), and Legner and Finson (1980). This class of models clearly represents the approach with the greatest potential for modeling flow systems in which the transport of turbulent quantities is important.

3. Algebraic Models. A disadvantage of the Reynolds stress transport approach is that multiple partial differential equations must be approximated. Some of these difficulties have been overcome by simplifying the exact transport equations to algebraic relationships. These approximations generally contain both the turbulent kinetic energy k and the isotropic dissipation ϵ as unknowns in addition to the Reynolds stresses. This approach therefore represents a level of complexity between the k - ϵ model and the Reynolds stress transport model. A recent application of this approach is presented by Noat and Rodi (1982) for steady-state channel flow. Although this type of model is more economical than the Reynolds stress transport models, it still lacks the potential universality of the transport models.

The above comparisons strongly indicate that the best approach for effectively dealing with flow systems as complicated as estuaries is the Reynolds stress transport model. The goal of this project, therefore, is to develop a two-dimensional (horizontal and vertical) computational model based on the complete turbulence equations that will address the specific application of tidally driven shallow-water wave flow regimes.

A theoretical examination of the governing equations, as they apply to the case of shallow-water wave propagation, is made to determine the dominant terms. This phase of development is based on the concept of scaling and nondimensionalization of the complete two-dimensional set of equations for turbulent flow. The rationale of this procedure is begun in Section 2 with a description of the parameters used to define shallow-water waves. This definition is presented to define the flow system for which the turbulent model is formulated. The mechanics of the scaling and nondimensionalization are presented in Sections 3 and 4. The end result of this procedure is an order of magnitude ranking of the individual terms in the governing equations. This is done by first defining the relative magnitude of the individual terms according to an exponential power of some small parameter ϵ . This initial grouping of terms of similar magnitude is further expanded based on the assumption that each flow variable can be expressed in terms of a power series expansion of the same

small parameter ϵ . This formal perturbation expansion technique demonstrates the relative magnitudes of the individual terms and provides valuable insight into the processes of turbulence. This insight is crucial for the following reasons: the equations governing turbulent flow contain more unknown quantities than equations. This indeterminate situation can be referred to as an open set of equations. In order to close the set, approximations for certain terms in the Reynolds stress transport equations must be made. These approximations are used so that the unknown terms, representing diffusion, pressure redistribution, and dissipation of turbulent energy, can be represented in terms of the known Reynolds stresses. The understanding of the relative magnitudes of these terms gained from the perturbation expansion analysis is therefore invaluable in the selection of an appropriate closure approximation. The selected closure models and the derivations of these models are presented in Section 5. The final closed set of equations representing the two-dimensional, shallow-water wave, turbulent flow system are presented in Section 6.

The application phase of this project is to develop a numerical model of the governing equations and to apply it to a flow case for which complete experimental data were available. A finite difference model was developed for this purpose. The numerical procedure used in this model and the associated boundary conditions necessary are presented in

Section 7. Verification of the model was made to the steady-state flume data collected by McQuivey in 1973. A description of this data and the results of the verification of the model to that data are presented in Section 8. In addition to the verification plots and comparisons, an analysis is made of the separate closure schemes used in the model. Finally, Section 9 discusses conclusions and future refinements and applications of the turbulence model.

SECTION 2

SHALLOW-WATER WAVES

The development of a turbulence model for shallow-water waves must be preceded by a description of the specific characteristics of those waves. It must also be shown that these characteristics are similar to those occurring in a tidally induced environment such as that found in a typical estuary. This definition is necessary for the development of nondimensionalization parameters that will be used extensively in the following sections.

In general, shallow-water waves (also referred to as long waves) represent that class of waves which have a wavelength much greater than the depth of fluid in which they are propagating. The implication of this statement is that the waves have a gradual slope. Tides, or astronomically induced waves, in coastal areas generally fall into this category since the depths of flow are usually small. Interest in the propagation of small amplitude waves was begun in the nineteenth century. Airy (1845) used the following parameter to characterize a long wave:

$$\frac{h}{L} \ll 1$$

2.1

where h is the depth of fluid and L is the distance between two consecutive nodes (wavelength). Using this definition and the assumption of hydrostatic pressure, Airy (1845) derived a set of equations which show that a wave cannot be propagated without a change in shape. It was later shown by Boussinesq (1871) and independently by Rayleigh (1876) that a permanent solitary long wave did, in fact, exist if the effects of dynamic pressures were accounted for. This contradiction of whether or not a long wave can propagate without change of shape led to the "long-wave paradox" disagreement reported by Lamb (1932) and later by Ursell (1953) and Madsen and Mei (1969).

Friedrichs (1948) presented a derivation of the governing equations which, while not giving a complete explanation of the paradox, began to shed considerable light on the problem. Friedrichs' (1948) derivation was based on a perturbation expansion in which the expansion parameter ϵ was similar to that defined by equation 2.1. He showed that the first approximation to the equations was consistent with the results of Airy, i.e. a wave of permanent form did not exist. A second order approximation of Friedrichs' derivation by Keller (1948), however, did show the existence of a wave of permanent shape. Even higher order approximations of the permanent wave were presented by Wehausen and Laitone (1960) by extending Friedrichs' approach.

The first satisfactory explanation to the long-wave paradox was presented by Ursell (1953). He demonstrated that the parameter

$$\epsilon = \frac{\zeta L^2}{h^3} \quad 2.2$$

where ζ is a measure of wave amplitude, should be taken into consideration in the derivation of long-wave theory as a "fundamental critical parameter." His derivations showed the following three cases indicating three types of long waves

Case 1: $\epsilon \ll 1$. The governing equation becomes linear with no permanent form solutions.

Case 2: $\epsilon = O(1)$. The governing equation were of the Boussinesq type and therefore showed permanent form solutions.

Case 3: $\epsilon \gg 1$. The governing equations become of the Airy type with no permanent form solution.

The above descriptions were presented so that the proper scaling parameters can be utilized for the non-dimensionalization of the governing equations for the application to estuarine conditions. At this point, typical values for an estuary should be defined. The following values are approximate averages for the Chesapeake Bay but are representative of other estuaries.

$$\begin{aligned} \zeta &\approx 2 \text{ ft} \\ L &\approx 200 \text{ miles} \\ h &\approx 30 \text{ ft} \end{aligned}$$

These values yield the following values for the expansion parameters shown in equations 2.1 and 2.2

$$\frac{h}{L} \approx 2.8 \times 10^{-5}$$

and

$$\frac{\zeta L^2}{h^3} \approx 8.3 \times 10^7$$

The above values demonstrate that typical estuarine systems are governed by Airy type equations. Consideration of this result led to the conclusion that the Friedrichs' expansion parameter should be a valid one for use in a tidally induced estuarine system. As previously mentioned, Wehausen and Laitone (1960) used a similar expansion parameter in their extensive expansion of the shallow-water wave equations for the case of inviscid, irrotational flow. Since this approach has shown to be valid for that case, it should be equally valid for the case of turbulent shallow-water wave propagation. The following two sections will therefore make use of Friedrichs' expansion parameter to develop the governing equations for viscous, rotational, turbulent shallow-water waves.

SECTION 3

THE REYNOLDS EQUATIONS AND THE CONTINUITY EQUATION

The governing equations for a constant density, incompressible, Newtonian fluid are the Navier-Stokes equations and the continuity equation. The turbulent equivalent of these relationships are the Reynolds equations and the turbulent continuity equation. The two-dimensional form can be written as follows:

Reynolds Equations

$$\frac{\partial \bar{U}}{\partial t} + \bar{U} \frac{\partial \bar{U}}{\partial x} + \bar{V} \frac{\partial \bar{V}}{\partial y} = - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \nu \left(\frac{\partial^2 \bar{U}}{\partial x^2} + \frac{\partial^2 \bar{U}}{\partial y^2} \right) - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{u'v}}{\partial y}$$

3.1

$$\frac{\partial \bar{V}}{\partial t} + \bar{U} \frac{\partial \bar{V}}{\partial x} + \bar{V} \frac{\partial \bar{V}}{\partial y} = - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial y} + \nu \left(\frac{\partial^2 \bar{V}}{\partial x^2} + \frac{\partial^2 \bar{V}}{\partial y^2} \right) - \frac{\partial \overline{u'v}}{\partial x} - \frac{\partial \overline{v'v'}}{\partial y}$$

3.2

Continuity Equation

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0$$

3.3

In addition to the above relationships, the boundary conditions for the free surface and bottom can be written as:

Surface Boundary Equation

$$\frac{\partial \bar{\zeta}}{\partial t} + \bar{U}_s \frac{\partial \bar{\zeta}}{\partial x} + \overline{u'_s \frac{\partial \zeta'}{\partial x}} - \bar{V}_s = 0 \quad 3.4$$

Bottom Boundary Equation

$$\bar{U}_B \frac{\partial \bar{H}}{\partial x} + \bar{V}_B = 0 \quad 3.5$$

In the above equations, the overbars represent the averaged value of either the mean flow quantity (in upper case) or the turbulent fluctuation (represented by the primed quantities). The individual terms are defined as

\bar{U} - mean horizontal velocity

\bar{V} - mean vertical velocity

\bar{P} - mean pressure

$\overline{u'u'}$, $\overline{u'v'}$, $\overline{v'v'}$ - Reynolds stresses (mean value of turbulent velocity fluctuation correlation)

$\bar{\zeta}$ - water-surface elevation above some datum

\bar{H} - depth of flow

x - primary horizontal axis

y - primary vertical axis

ν - kinematic viscosity

A complete derivation of equations 3.1 through 3.5 is presented in appendices A and B.

In order to begin the order of magnitude analysis, it is necessary to nondimensionalize the governing equations such that the horizontal and vertical distances are stretched

by different amounts. The following nondimensional terms are therefore defined:

$$\hat{U} = \frac{\bar{U}}{\sqrt{gh}} = \frac{\bar{U}}{V_o} \quad 3.6$$

$$\hat{V} = \frac{\bar{V}}{\sqrt{gh}} = \frac{\bar{V}}{V_o} \quad 3.7$$

$$\hat{x} = \frac{x}{l} \quad 3.8$$

$$\hat{y} = \frac{y}{h} \quad 3.9$$

$$\hat{t} = \frac{\sqrt{gh}}{l} t = \frac{V_o t}{l} \quad 3.10$$

$$\hat{p} = \frac{\bar{p}}{\rho gh} = \frac{\bar{p}}{\rho V_o^2} \quad 3.11$$

$$\hat{\zeta} = \frac{\bar{\zeta}}{h} \quad 3.12$$

$$\hat{H} = \frac{\bar{H}}{h} \quad 3.13$$

The above definitions make use of the Friedrichs' expansion parameter such that

$$\epsilon = \frac{h}{l} \ll 1$$

where h is a length scale representing depth of flow and l represents a wavelength scale. The reference velocity V_o represents the velocity of a gravity wave in a fluid of depth h . The turbulent fluctuation terms appearing in equations 3.1, 3.2, and 3.4 are nondimensionalized by use

of an as yet undetermined coefficient. These relationships are written as follows:

$$\hat{u}'_s \frac{\partial \zeta'}{\partial x} = \overline{u'_s \frac{\partial \zeta'}{\partial x}} / \gamma \quad 3.14$$

and

$$\hat{u}'_i \hat{u}'_j = \overline{u'_i u'_j} / \alpha \quad 3.15$$

where indicial notation has been used for the Reynolds stresses $\overline{u'u'}$, $\overline{u'v'}$, and $\overline{v'v'}$. A reference Reynolds number can also be defined based on the reference velocity.

$$R = \frac{V_o h}{\nu} \quad 3.16$$

Substitution of the nondimensional relationships of 3.6 - 3.15 into equations 3.1 - 3.5 and grouping according to powers of the expansion coefficient $\epsilon = h/l$ yields the following equations:

$$\begin{aligned} \epsilon^0 \left(\hat{V} \frac{\partial \hat{U}}{\partial y} - \frac{1}{R} \frac{\partial^2 \hat{U}}{\partial y^2} \right) + \epsilon \left(\frac{\partial \hat{U}}{\partial t} + \hat{U} \frac{\partial \hat{U}}{\partial x} + \frac{\partial \hat{P}}{\partial x} \right) - \epsilon^2 \left(\frac{1}{R} \frac{\partial^2 \hat{U}}{\partial x^2} \right) \\ + \epsilon \left(\frac{\alpha}{V_o^2} \frac{\partial \hat{u}'_i \hat{u}'_i}{\partial x} \right) + \epsilon^0 \left(\frac{\alpha}{V_o^2} \frac{\partial \hat{u}'_i \hat{v}'_i}{\partial y} \right) = 0 \end{aligned} \quad 3.17$$

$$\begin{aligned}
& \epsilon^0 \left(\hat{V} \frac{\partial \hat{V}}{\partial y} + \frac{\partial \hat{P}}{\partial y} + 1 - \frac{1}{R} \frac{\partial^2 \hat{V}}{\partial y^2} \right) + \epsilon \left(\frac{\partial \hat{V}}{\partial t} + \hat{U} \frac{\partial \hat{V}}{\partial x} \right) - \epsilon^2 \left(\frac{1}{R} \frac{\partial^2 \hat{V}}{\partial x^2} \right) \\
& + \epsilon \left(\frac{\alpha}{V_0^2} \frac{\partial u' v'}{\partial x} \right) + \epsilon^0 \left(\frac{\alpha}{V_0^2} \frac{\partial v' v'}{\partial y} \right) = 0
\end{aligned} \tag{3.18}$$

$$\epsilon \left(\frac{\partial U}{\partial x} \right) + \epsilon^0 \left(\frac{\partial \hat{V}}{\partial y} \right) = 0 \tag{3.19}$$

$$\epsilon \left(\frac{\partial \zeta}{\partial t} + \hat{U}_s \frac{\partial \zeta}{\partial x} \right) + \epsilon^0 \left(\frac{\gamma}{V_0^2} u'_s \frac{\partial \zeta'}{\partial x} - \hat{V}_s \right) = 0 \tag{3.20}$$

$$\epsilon \left(\frac{\partial \hat{H}}{\partial x} \right) + \hat{V}_B = 0 \tag{3.21}$$

An analysis of this preliminary grouping leads to some simplifications which can be used in the further expansions of the terms. For example, the following conclusions can be deduced:

a) equation 3.19 shows that

$$\frac{\partial \hat{V}}{\partial y} = 0(\epsilon)$$

or that the vertical velocity \hat{V} is essentially constant throughout the water column. Equation 3.21, however, shows that the bottom velocity is small; for example:

$$\hat{V}_B = 0(\epsilon)$$

It can then be seen that the constant vertical velocity is small and can be expressed as shown below

$$\hat{V} = o(\epsilon)$$

b) The as yet undefined coefficient α must be selected such that the final set of equations will accurately represent the complete range (manifold) of solutions to the given problem of small amplitude waves. An extreme case can be presented as steady uniform flow in the \hat{x} - direction. For this case, equation 3.17 can be written as

$$\epsilon^0 \left(V \frac{\partial \hat{U}}{\partial y} - \frac{1}{R} \frac{\partial^2 \hat{U}}{\partial y^2} \right) + \epsilon \left(\frac{\partial \hat{P}}{\partial x} \right) + \epsilon^0 \left(\frac{\alpha}{V_0^2} \frac{\partial \hat{u}' \hat{v}'}{\partial y} \right) = 0$$

Two conditions can now be considered. First, if the vertical velocity \hat{V} is assumed zero and the flow is considered nonturbulent ($\hat{u}' \hat{v}' = 0$), then it can be seen that the above relationship should be written as

$$\epsilon \left(- \frac{1}{\epsilon R} \frac{\partial^2 \hat{U}}{\partial y^2} + \frac{\partial \hat{P}}{\partial x} \right) = 0$$

which is the equation for laminar flow between parallel plates (Daily and Harleman, 1966, p.116). If, on the other hand, turbulence is considered, it can be seen that the following relationship must be valid

$$\epsilon^0 \left(\frac{\alpha}{V_0^2} \frac{\partial \hat{u}' \hat{v}'}{\partial y} \right) = 0 \quad (\epsilon)$$

This leads to the definition of the Reynolds stress coefficient as

$$\alpha = \epsilon V_o^2 \quad 3.22$$

Equations 3.17 through 3.19 can now be written entirely in terms of ϵ as shown below.

$$\begin{aligned} \epsilon^0 \left(\hat{V} \frac{\partial \hat{u}}{\partial y} \right) + \epsilon \left(\frac{\partial \hat{U}}{\partial t} + \hat{U} \frac{\partial \hat{U}}{\partial x} + \frac{\partial \hat{P}}{\partial x} - \frac{1}{\epsilon R} \frac{\partial^2 \hat{U}}{\partial y^2} + \frac{\partial \hat{u}' v'}{\partial y} \right) \\ + \epsilon^2 \left(\frac{\partial \hat{u}' u'}{\partial x} - \frac{1}{\epsilon R} \frac{\partial^2 \hat{U}}{\partial x^2} \right) = 0 \end{aligned} \quad 3.23$$

$$\begin{aligned} \epsilon^0 \left(\hat{V} \frac{\partial \hat{V}}{\partial y} + \frac{\partial \hat{P}}{\partial y} + 1 \right) + \epsilon \left(\frac{\partial \hat{V}}{\partial t} + \hat{U} \frac{\partial \hat{V}}{\partial x} - \frac{1}{\epsilon R} \frac{\partial^2 \hat{V}}{\partial y^2} + \frac{\partial \hat{v}' v'}{\partial y} \right) \\ + \epsilon^2 \left(\frac{\partial \hat{u}' v'}{\partial x} \right) - \epsilon^3 \left(\frac{1}{\epsilon R} \frac{\partial^2 \hat{V}}{\partial x^2} \right) = 0 \end{aligned} \quad 3.24$$

$$\epsilon \left(\frac{\partial \hat{U}}{\partial x} \right) + \frac{\partial \hat{V}}{\partial y} = 0 \quad 3.25$$

The above equations now represent the initial non-dimensionalization of the governing equations. The boundary condition equations 3.20 and 3.21 will be considered at a later point since they do not affect the further development of the Reynolds equations or the continuity equation.

In order to complete the order of magnitude analysis of equations 3.1 - 3.3, it is necessary to assume that the variable quantities can be represented in terms of a power series expansion with respect to some small parameter, in this case the expansion parameter ϵ (Stoker, 1957, and Ippen, 1966). This identical procedure was used by Wehausen and Laitone (1960) in their derivation of the nonturbulent, inviscid shallow-water wave. Following the above examples, the variable expansions take the following form:

$$\hat{U} = V_0 + \epsilon U_1 + \epsilon^2 U_2 + \dots \epsilon^m U_m$$

$$\hat{V} = \epsilon V_1 + \epsilon^2 V_2 + \dots \epsilon^m V_m \quad (\text{since } \hat{V} = 0 \text{ } (\epsilon)) \quad 3.26$$

$$\hat{P} = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \dots \epsilon^m P_m$$

$$\hat{u'_i u'_j} = (u'_i u'_j)_0 + \epsilon (u'_i u'_j)_1 + \epsilon^2 (u'_i u'_j)_2 + \dots \epsilon^m (u'_i u'_j)_m$$

Substitution of equations 3.26 into equations 3.23 through 3.25 and collection according to powers of ϵ give the following order of magnitude relationships (for brevity, only the relationships through $O(\epsilon^3)$ will be written. The overmarks indicating nondimensional x and y directions will also be omitted):

$$\begin{aligned}
& \epsilon \left[\frac{\partial U_0}{\partial t} + U_0 \frac{\partial U_0}{\partial x} + V_1 \frac{\partial U_0}{\partial y} + \frac{\partial P_0}{\partial x} - \frac{1}{\epsilon R} \frac{\partial^2 U_0}{\partial y^2} + \frac{\partial (u'v')_0}{\partial y} \right] \\
& + \epsilon^2 \left[\frac{\partial U_1}{\partial t} + U_0 \frac{\partial U_1}{\partial x} + U_1 \frac{\partial U_0}{\partial x} + V_1 \frac{\partial U_1}{\partial y} + V_2 \frac{\partial U_0}{\partial y} + \frac{\partial P_1}{\partial x} \right. \\
& \quad \left. - \frac{1}{\epsilon R} \frac{\partial^2 U_1}{\partial y^2} + \frac{\partial (u'v')_1}{\partial y} + \frac{\partial (u'u')_0}{\partial x} \right] \\
& + \epsilon^3 \left[\frac{\partial U_2}{\partial t} + U_0 \frac{\partial U_2}{\partial x} + U_1 \frac{\partial U_1}{\partial x} + U_2 \frac{\partial U_0}{\partial x} + V_1 \frac{\partial U_2}{\partial y} + V_2 \frac{\partial U_1}{\partial y} \right. \\
& \quad \left. + \frac{\partial P_2}{\partial x} - \frac{1}{\epsilon R} \frac{\partial^2 U_2}{\partial y^2} + \frac{\partial (u'v')_2}{\partial y} + \frac{\partial (u'u')_1}{\partial x} - \frac{1}{\epsilon R} \frac{\partial^2 U_0}{\partial x^2} \right] \\
& + 0 (\epsilon^4) = 0
\end{aligned}$$

3.27

$$\begin{aligned}
& \epsilon^0 \left[\frac{\partial P_0}{\partial y} + 1 \right] \\
& + \epsilon \left[\frac{\partial P_1}{\partial y} + \frac{\partial (v'v')_0}{\partial y} \right] \\
& + \epsilon^2 \left[\frac{\partial V_1}{\partial t} + U_0 \frac{\partial V_1}{\partial x} + V_1 \frac{\partial V_1}{\partial y} + \frac{\partial P_2}{\partial y} - \frac{1}{\epsilon R} \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial (v'v')_1}{\partial y} \right. \\
& \quad \left. + \frac{\partial (u'v')_0}{\partial x} \right] + \epsilon^3 \left[\frac{\partial V_2}{\partial t} + U_1 \frac{\partial V_1}{\partial x} + U_0 \frac{\partial V_2}{\partial x} + V_1 \frac{\partial V_2}{\partial y} + V_2 \frac{\partial V_1}{\partial y} \right. \\
& \quad \left. - \frac{1}{\epsilon R} \frac{\partial^2 V_2}{\partial y^2} + \frac{\partial (v'v')_2}{\partial y} + \frac{\partial (u'v')_1}{\partial x} \right] + 0 (\epsilon^4) = 0
\end{aligned}$$

3.28

$$\epsilon \left[\frac{\partial U_0}{\partial x} + \frac{\partial V_1}{\partial y} \right] + \epsilon^2 \left[\frac{\partial U_1}{\partial x} + \frac{\partial V_2}{\partial y} \right] + \epsilon^3 \left[\frac{\partial U_2}{\partial x} + \frac{\partial V_3}{\partial y} \right] + 0 (\epsilon^4) = 0 \quad 3.29$$

Inspection of the order of magnitude relationships presented in equations 3.27 through 3.29 demonstrate the following:

- a) The zero order solution of the Reynolds equations is that of hydrostatic pressure. Although this is a common assumption, the method of analysis does confirm the relationship.
- b) The Reynolds stress terms are of the same order of magnitude as the terms representing convection, pressure, viscosity, etc. The proper modeling of a turbulent flow system then requires the proper modeling of the Reynolds stresses.
- c) The system of equations does not represent a closed set of equations; they now contain the unknown Reynolds stresses $\hat{u}_i' u_j'$.

Conclusions a) and b) are evident from the presented analysis; however, the approach to conclusion c) represents the difference between proper turbulence modeling and "pseudo" turbulence modeling. The common procedure at this point is to represent the Reynolds stresses through some

simplified closing relationship using parameters such as mixing lengths or combinations of mixing length and turbulent kinetic energy. Early models of this type have been developed by Prandtl (1925,1945) and Kolmogorov (1942). Many of the more recent closure techniques are presented by Launder and Spalding (1972) and by Rodi (1980). The approach selected for this analysis is the approximation of the Reynolds stresses based on the exact equations for the Reynolds stresses (Reynolds stress transport equations). Although these equations also do not represent a closed set of equations, closure at this level represents a higher order of accuracy turbulence model known as a second-order closure model. The closure approximations employed for the model are presented in Section 5 following a detailed analysis of the Reynolds stress transport equations in the following section. This is done in an analagous manner to that in which the perturbated Reynolds equations of this section were developed.

SECTION 4

THE REYNOLDS STRESS TRANSPORT EQUATIONS

The governing equations for the transport of the turbulent Reynolds stresses for a constant density, incompressible, Newtonian fluid can be written in indicial notation as follows:

$$\begin{aligned}
 & \frac{\partial \overline{u'_i u'_j}}{\partial t} + \overline{u'_j u'_k} \frac{\partial \overline{u'_i}}{\partial x_k} + \overline{u'_i u'_k} \frac{\partial \overline{u'_j}}{\partial x_k} + \overline{u_k} \frac{\partial \overline{u'_i u'_j}}{\partial x_k} \\
 &= \frac{\overline{p'}}{\rho} \frac{\partial \overline{u'_j}}{\partial x_i} + \frac{\overline{p'}}{\rho} \frac{\partial \overline{u'_i}}{\partial x_j} - \frac{1}{\rho} \left(\frac{\partial \overline{p' u'_j}}{\partial x_i} + \frac{\partial \overline{p' u'_i}}{\partial x_j} \right) \\
 &+ \nu \left(\frac{\partial^2 \overline{u'_i u'_j}}{\partial x_k^2} - 2 \frac{\partial \overline{u'_i}}{\partial x_k} \frac{\partial \overline{u'_j}}{\partial x_k} \right) - \frac{\partial \overline{u'_i u'_j u'_k}}{\partial x_k}
 \end{aligned} \tag{4.1}$$

The overbarred quantities represent the averaged values of either the mean flow variables (in upper case letters) or the fluctuating variables (represented by the primes). The above equations are derived through the manipulation of the Reynolds equations and the continuity equation. A complete derivation of equation 4.1 is shown in Appendix C.

Equation 4.1 represents six equations for the Reynolds stresses $\overline{u' u'}$, $\overline{v' v'}$, $\overline{w' w'}$, $\overline{u' v'}$, $\overline{u' w'}$, and $\overline{v' w'}$. The case of two-dimensional flow (for which $\overline{u_1} = f(x_2)$, $\overline{u_2} = \overline{u_3} = 0$

and symmetry of flow with respect to planes perpendicular to the x_3 axis) reduces the number of equations to four ($\overline{u'u'}$, $\overline{v'v'}$, $\overline{w'w'}$, and $\overline{u'v'}$). This is due to the fact that all correlations involving u'_3 and all uneven derivatives with respect to the x_3 direction are zero (Hinze, 1959, p.252, and Cebeci and Smith, 1974, p.83). These four equations now represent the governing equations for the transport of the Reynolds stresses for the two-dimensional flow case. Each equation can now be nondimensionalized in a similar manner to that performed for the Reynolds equations in the previous section.

The nondimensionalization process begins with the use of equations 3.6 through 3.10. The z - axis direction, now represented in equation 4.1, is scaled identically to the x - axis as shown below

$$\hat{z} = z/l \quad 4.2$$

Use will also be made of the results shown by equation 3.22 for the Reynolds stress scale. This was shown to be

$$\hat{u}'_i \hat{u}'_j = \overline{u'_i u'_j} / \alpha$$

or

$$\hat{u}'_i \hat{u}'_j = \overline{u'_i u'_j} / \epsilon v_o^2$$

In addition to the above parameters, it is also necessary to develop scaling relationships for the new correlation quantities in equation 4.1 which represent pressure

redistribution, dissipation, and diffusion of turbulence. Since each of these terms are of unknown relative magnitude, they will be addressed separately.

- a) Diffusion. Scaling for the triple correlation term representing diffusion of turbulence is based on the assumption that a single fluctuation can be written as follows:

$$\hat{u}_i' = \frac{u_i'}{v_o \sqrt{\epsilon}} \quad 4.3$$

This represents an extension of equation 3.22. Use of this relationship yields the following nondimensional form for the diffusion term

$$u_i' \hat{u}_j' u_k' = \overline{u_i' u_j' u_k'} / \epsilon^{3/2} v_o^3 \quad 4.4$$

- b) Pressure Redistribution. The terms in equation 4.1 which contain pressure fluctuation correlations can be consolidated as follows:

$$\overline{u_j' \frac{\partial p'}{\partial x_i}} = \overline{p \frac{\partial u_j'}{\partial x_i}} - \frac{\partial \overline{p' u_j'}}{\partial x_i}$$

An as yet undetermined coefficient can now be introduced as

$$\hat{u}_j' \frac{p'}{x_i} = \frac{1}{\rho} \overline{u_j' \frac{\partial p'}{\partial x_i}} / \frac{\epsilon v_o^3}{x_i - \text{scale}}$$

where the term $x_i - \text{scale}$ refers to the appropriate x, y, or z direction scale and use has

been made of equation 4.3. A Poisson equation for the single pressure fluctuation can be written as follows

$$-\frac{1}{\rho} \frac{\partial}{\partial x_k} \left(\frac{\partial p'}{\partial x_k} \right) = \frac{\partial^2}{\partial x_i \partial x_j} (\bar{U}_j u'_i + \bar{U}_i u'_j + u'_i u'_j)$$

The derivation of this relationship is based on the divergence of the Navier-Stokes equations and is presented in Appendix D. If the spacial derivatives are neglected, it can be seen that the lowest order of magnitude representation of the pressure fluctuation can be written as follows:

$$-\frac{p'}{\rho} \approx 0 (\bar{U} u')$$

The term of interest must then be expressible as

$$\frac{1}{\rho} u'_j \frac{\partial p'}{\partial x_i} = 0 \left(\bar{U} \frac{\partial u'_i u'_j}{\partial x_i} \right) = 0 \left(u'_i u'_j \frac{\partial U}{\partial x_i} \right)$$

This assumption leads to the assumption that

$$\theta = \epsilon^{1/2} V_o^2$$

or that the scaled term can be written as

$$\hat{u'_j} \frac{\partial p'}{\partial x_i} = \frac{1}{\rho} \overline{u'_j \frac{\partial p'}{\partial x_i}} / \frac{\epsilon V_o^3}{x_i - \text{scale}}$$

A comment can now be made concerning the term labeled $x_i - \text{scale}$. The pressure fluctuation

term represents the sum of the following two parts:

$$u'_j \frac{\partial p'}{\partial x_i} = \frac{\partial u'_j p'}{\partial x_i} - p' \frac{\partial u'_j}{\partial x_i}$$

It is generally acknowledged that the first term on the right is considerably smaller in magnitude than the second term (Hanjalic and Launder, 1972). In this case, the scaling must be based on careful examination of that term. For example, in the equation for $\overline{u'u'}$, $\overline{v'v'}$, and $\overline{w'w'}$, the second term becomes

$$\overline{p' \frac{\partial u'}{\partial x}}, \overline{p' \frac{\partial v'}{\partial y}}, \text{ or } \overline{p' \frac{\partial w'}{\partial z}}$$

which are all interrelated due to the continuity equation. This would appear reasonable since pressure exerts in all directions equally. With this analogy in mind, the x_i - scale was selected as that of the y - direction. Therefore, the final scaling for the pressure redistribution term is as shown below

$$\hat{u'_j \frac{\partial p'}{\partial x_i}} = \frac{1}{\rho} \overline{u'_j \frac{\partial p'}{\partial x_i}} / \frac{\epsilon v_o^2}{h} \quad 4.5$$

c) Dissipation. The terms in equation 4.1 which represent dissipation of turbulence can initially be determined following the expansion of equation

4.1 based on the already determined coefficients.

This temporary relationship is as follows:

$$\frac{\partial \hat{u}_i'}{\partial x_k} \frac{\partial \hat{u}_j'}{\partial x_k} = \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} / \frac{\epsilon V_o^2 \beta}{(x_k - \text{scale})^2} \quad 4.6$$

The individual equations representing the transport of the Reynolds stresses can now be written and nondimensionalized by use of equations 3.6 - 3.10, 4.2, and 4.4 - 4.6. The dimensional form (presented for reference purposes) and the nondimensional form are shown as follows:

$\overline{u'u'}$:

$$\begin{aligned} & \frac{\partial \overline{u'u'}}{\partial t} + 2 \overline{u'u'} \frac{\partial \bar{U}}{\partial x} + 2 \overline{u'v'} \frac{\partial \bar{U}}{\partial y} + \bar{U} \frac{\partial \overline{u'u'}}{\partial x} + \bar{V} \frac{\partial \overline{u'u'}}{\partial y} \\ & + \frac{\partial \overline{u'u'u'}}{\partial x} + \frac{\partial \overline{u'u'v'}}{\partial y} + \frac{2}{\rho} \overline{u' \frac{\partial p'}{\partial x}} - \nu \left(\frac{\partial^2 \overline{u'u'}}{\partial x^2} + \frac{\partial^2 \overline{u'u'}}{\partial y^2} \right) \\ & + \frac{\partial \overline{u'}}{\partial x} \frac{\partial \overline{u'}}{\partial x} + 2\nu \frac{\partial \overline{u'}}{\partial y} \frac{\partial \overline{u'}}{\partial y} + 2\nu \frac{\partial \overline{u'}}{\partial z} \frac{\partial \overline{u'}}{\partial z} = 0 \end{aligned} \quad 4.7$$

$\hat{u}'u'$:

$$\begin{aligned} & \epsilon \left(\frac{\partial \hat{u}'u'}{\partial t} + 2\hat{u}'u' \frac{\partial \hat{U}}{\partial x} + \hat{U} \frac{\partial \hat{u}'u'}{\partial x} \right) + 2 \hat{u}'v' \frac{\partial \hat{U}}{\partial y} + \hat{V} \frac{\partial \hat{u}'u'}{\partial y} \\ & + \epsilon^{3/2} \left(\frac{\partial \hat{u}'\hat{u}'u'}{\partial x} \right) + \epsilon^{1/2} \left(\frac{\partial \hat{u}'\hat{u}'v'}{\partial y} \right) + 2\hat{u}' \frac{\partial \hat{p}'}{\partial x} \end{aligned}$$

$$\begin{aligned}
& - \varepsilon \left(\frac{\nu}{V_0 l} \right) \frac{\partial^2 \hat{u}' u'}{\partial \hat{x}^2} - \left(\frac{\nu}{V_0 h} \right) \frac{\partial^2 \hat{u}' u'}{\partial \hat{y}^2} + \varepsilon \left(\frac{2\beta \nu}{V_0 l} \right) \frac{\partial \hat{u}'}{\partial \hat{x}} \frac{\partial \hat{u}'}{\partial \hat{x}} \\
& + \left(\frac{2\beta \nu}{V_0 h} \right) \frac{\partial \hat{u}'}{\partial \hat{y}} \frac{\partial \hat{u}'}{\partial \hat{y}} + \varepsilon \left(\frac{2\beta \nu}{V_0 l} \right) \frac{\partial \hat{u}'}{\partial \hat{z}} \frac{\partial \hat{u}'}{\partial \hat{z}} = 0
\end{aligned} \tag{4.8}$$

$\overline{v'v'}$:

$$\begin{aligned}
& \frac{\partial \overline{v'v'}}{\partial t} + 2 \overline{u'v'} \frac{\partial \overline{v}}{\partial x} + 2 \overline{v'v'} \frac{\partial \overline{v}}{\partial y} + \overline{u} \frac{\partial \overline{v'v'}}{\partial x} + \overline{v} \frac{\partial \overline{v'v'}}{\partial y} \\
& + \frac{\partial \overline{u'v'v'}}{\partial x} + \frac{\partial \overline{v'v'v'}}{\partial y} + \frac{2}{\rho} \overline{v'} \frac{\partial \overline{p'}}{\partial y} - \nu \left(\frac{\partial^2 \overline{v'v'}}{\partial x^2} + \frac{\partial^2 \overline{v'v'}}{\partial y^2} \right) \\
& + 2\nu \frac{\partial \overline{v'}}{\partial x} \frac{\partial \overline{v'}}{\partial x} + 2\nu \frac{\partial \overline{v'}}{\partial y} \frac{\partial \overline{v'}}{\partial y} + 2\nu \frac{\partial \overline{v'}}{\partial z} \frac{\partial \overline{v'}}{\partial z} = 0
\end{aligned} \tag{4.9}$$

$\hat{v'v'}$:

$$\begin{aligned}
& \varepsilon \left(\frac{\partial \hat{v'v'}}{\partial t} + 2 \hat{u}' \hat{u}' \frac{\partial \hat{v}}{\partial x} + \hat{u} \frac{\partial \hat{v'v'}}{\partial x} \right) + 2 \hat{v'v'} \frac{\partial \hat{v}}{\partial y} + \frac{\hat{v'v'}}{\partial y} \\
& + \varepsilon^{3/2} \left(\frac{\partial \hat{u}' \hat{v'v'}}{\partial x} \right) + \varepsilon^{1/2} \frac{\partial \hat{v'v'v'}}{\partial y} + 2 \hat{v'} \frac{\partial \hat{p'}}{\partial y} \\
& - \varepsilon \left(\frac{\nu}{V_0 l} \right) \frac{\partial^2 \hat{v'v'}}{\partial \hat{x}^2} - \left(\frac{\nu}{V_0 h} \right) \frac{\partial^2 \hat{v'v'}}{\partial \hat{y}^2} + \varepsilon \left(\frac{2\nu\beta}{V_0 l} \right) \frac{\partial \hat{v'}}{\partial \hat{x}} \frac{\partial \hat{v'}}{\partial \hat{x}} \\
& + \left(\frac{2\nu\beta}{V_0 h} \right) \frac{\partial \hat{v'}}{\partial \hat{y}} \frac{\partial \hat{v'}}{\partial \hat{y}} + \varepsilon \left(\frac{2\nu\beta}{V_0 l} \right) \frac{\partial \hat{v'}}{\partial \hat{z}} \frac{\partial \hat{v'}}{\partial \hat{z}} = 0
\end{aligned} \tag{4.10}$$

$\overline{w'w'}:$

$$\begin{aligned} \frac{\partial \overline{w'w'}}{\partial t} + \bar{U} \frac{\partial \overline{w'w'}}{\partial x} + \bar{V} \frac{\partial \overline{w'w'}}{\partial y} + \frac{\partial \overline{u'w'w'}}{\partial x} + \frac{\partial \overline{v'w'w'}}{\partial y} + \frac{2}{\rho} \overline{w' \frac{\partial p'}{\partial z}} \\ + 2 \frac{\partial \overline{w'w'}}{\partial x} \frac{\partial \overline{w'}}{\partial x} + 2v \frac{\partial \overline{w'w'}}{\partial y} \frac{\partial \overline{w'}}{\partial y} + 2v \frac{\partial \overline{w'w'}}{\partial z} \frac{\partial \overline{w'}}{\partial z} = 0 \end{aligned} \quad 4.11$$

$\hat{w'w'}:$

$$\begin{aligned} \epsilon \left(\frac{\partial \hat{w'w'}}{\partial t} + \hat{U} \frac{\partial \hat{w'w'}}{\partial x} \right) + \hat{V} \frac{\partial \hat{w'w'}}{\partial y} + \epsilon^{3/2} \frac{\partial \hat{u'w'w'}}{\partial x} + \epsilon^{1/2} \frac{\partial \hat{v'w'w'}}{\partial y} \\ + 2\hat{w'} \frac{\partial \hat{p'}}{\partial z} + \epsilon \left(\frac{2v\beta}{V_{o1}} \right) \frac{\partial \hat{w'}}{\partial x} \frac{\partial \hat{w'}}{\partial x} + \left(\frac{2v\beta}{V_{oh}} \right) \frac{\partial \hat{w'}}{\partial y} \frac{\partial \hat{w'}}{\partial y} \\ + \epsilon \left(\frac{2v\beta}{V_{o1}} \right) \frac{\partial \hat{w'}}{\partial z} \frac{\partial \hat{w'}}{\partial z} = 0 \end{aligned} \quad 4.12$$

$\overline{u'v'}:$

$$\begin{aligned} \frac{\partial \overline{u'v'}}{\partial t} + \overline{u'u'} \frac{\partial \bar{U}}{\partial x} + \overline{v'v'} \frac{\partial \bar{U}}{\partial y} + \overline{u'u'} \frac{\partial \bar{V}}{\partial x} + \overline{u'v'} \frac{\partial \bar{V}}{\partial y} + \bar{U} \frac{\partial \overline{u'v'}}{\partial x} \\ + \bar{V} \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'u'v'}}{\partial x} + \frac{\partial \overline{u'v'v'}}{\partial y} + \frac{1}{\rho} \overline{v' \frac{\partial p'}{\partial x}} + \frac{1}{\rho} \overline{u' \frac{\partial p'}{\partial y}} \\ - v \frac{\partial^2 \overline{u'v'}}{\partial x^2} - v \frac{\partial^2 \overline{u'v'}}{\partial y^2} + 2v \frac{\partial \overline{u'}}{\partial x} \frac{\partial \overline{v'}}{\partial x} + 2v \frac{\partial \overline{u'}}{\partial y} \frac{\partial \overline{v'}}{\partial y} \\ + 2v \frac{\partial \overline{u'}}{\partial z} \frac{\partial \overline{v'}}{\partial z} = 0 \end{aligned} \quad 4.13$$

$u'v'$:

$$\begin{aligned}
& \epsilon \left(\frac{\partial \hat{u}'v'}{\partial \hat{t}} + u'v' \frac{\partial \hat{U}}{\partial \hat{x}} + u'u' \frac{\partial \hat{V}}{\partial \hat{x}} + \hat{U} \frac{\partial \hat{u}'v'}{\partial \hat{x}} \right) + v'v' \frac{\partial \hat{U}}{\partial \hat{y}} \\
& + u'v' \frac{\partial \hat{V}}{\partial \hat{y}} + \hat{V} \frac{\partial \hat{u}'v'}{\partial \hat{y}} + \epsilon^{3/2} \frac{\partial \hat{u}'\hat{v}'v'}{\partial \hat{x}} + \epsilon^{1/2} \frac{\partial \hat{u}'\hat{v}'v'}{\partial \hat{y}} \\
& + v' \frac{\partial \hat{p}'}{\partial \hat{x}} + u' \frac{\partial \hat{p}'}{\partial \hat{y}} - \epsilon \left(\frac{v}{V_o l} \right) \frac{\partial^2 \hat{u}'v'}{\partial \hat{x}^2} - \left(\frac{v}{V_o h} \right) \frac{\partial^2 \hat{u}'v'}{\partial \hat{y}^2} \\
& + \epsilon \left(\frac{2v\beta}{V_o l} \right) \frac{\partial \hat{u}'}{\partial \hat{x}} \frac{\partial \hat{v}'}{\partial \hat{x}} + \left(\frac{2v\beta}{V_o h} \right) \frac{\partial \hat{u}'}{\partial \hat{y}} \frac{\partial \hat{v}'}{\partial \hat{y}} + \epsilon \left(\frac{2v\beta}{V_o l} \right) \frac{\partial \hat{u}'}{\partial \hat{z}} \frac{\partial \hat{v}'}{\partial \hat{z}} = 0
\end{aligned}$$

4.14

The final remaining nondimensionalizing coefficient β can be determined by inspection of the zero order terms of equation 4.10. The sum of these terms must be at most of the order $\epsilon^{1/2}$. This relationship is shown below

$$\begin{aligned}
& \epsilon^0 \left[2v'v' \frac{\partial \hat{V}}{\partial \hat{y}} + \hat{V} \frac{\partial \hat{v}'v'}{\partial \hat{y}} - \frac{1}{R} \frac{\partial^2 \hat{v}'v'}{\partial \hat{y}^2} + \frac{2\beta}{R} \frac{\partial \hat{v}'}{\partial \hat{y}} \frac{\partial \hat{v}'}{\partial \hat{y}} + \partial \hat{v}' \frac{\partial \hat{p}'}{\partial \hat{y}} \right] \\
& = 0 \quad (\epsilon^{1/2})
\end{aligned}$$

From previous analyses, it has been shown that:

- a) The first two terms are of order ϵ since it has been determined that $\hat{V} = 0 \quad (\epsilon)$.
- b) The term $\frac{1}{R} \frac{\partial^2 \hat{v}'v'}{\partial \hat{y}^2}$ is of order ϵ since it has been shown in Section 3 that the term $1/R$ is small and should be expressed in the form $\epsilon \left(\frac{1}{\epsilon R} \right)$ for proper scaling.

- c) The pressure fluctuation terms are approximately of the order of magnitude of $u_i' u_j' \frac{\partial \hat{U}}{\partial x_i}$ and are therefore not insignificant in value.

In view of the above observations, it is reasonable to assume that the last two terms are of equivalent orders of magnitude for the following reason: The pressure fluctuation term represents a source of turbulent energy transferred into the system. This source of energy must be balanced by the dissipation term since the remaining terms have been shown to be small. The following relationship is then based on this assumption

$$v' \frac{\partial p'}{\partial \hat{y}} \approx u' v' \frac{\partial \hat{U}}{\partial \hat{y}} \approx \frac{\beta}{R} \frac{\partial v'}{\partial \hat{y}} \frac{\partial v'}{\partial \hat{y}}$$

In order for this relationship to occur, the term β/R must be of order unity. Therefore, the final scaling parameter is

$$\beta = \frac{1}{\epsilon}$$

Equations 4.8, 4.10, 4.12, and 4.14 can now be written in a manner compatible with equations 3.23 through 3.25. The parameter $1/\epsilon R$ is used in the appropriate locations similar to the earlier derivations. The following relationships result $u' u'$:

$$\begin{aligned} & \epsilon^0 (2u' \hat{v}' \frac{\partial \hat{U}}{\partial \hat{y}} + \hat{v}' \frac{\partial u' u'}{\partial \hat{y}} + \frac{2}{\epsilon R} \frac{\partial u'}{\partial \hat{y}} \frac{\partial u'}{\partial \hat{y}} + 2u' \frac{\partial p'}{\partial \hat{x}}) + \epsilon^{1/2} (\frac{\partial u' \hat{u}' v'}{\partial \hat{y}}) \\ & + \epsilon (\frac{\partial u' \hat{u}'}{\partial \hat{t}} + 2u' \hat{u}' \frac{\partial \hat{U}}{\partial \hat{x}} + \hat{U} \frac{\partial u' \hat{u}'}{\partial \hat{x}} - \frac{1}{\epsilon R} \frac{\partial^2 u' \hat{u}'}{\partial \hat{y}^2}) + \epsilon^{3/2} \frac{\partial u' \hat{u}' u'}{\partial \hat{x}} \\ & + \epsilon^2 (\frac{2}{\epsilon R} \frac{\partial u'}{\partial \hat{x}} \frac{\partial u'}{\partial \hat{x}} + \frac{2}{\epsilon R} \frac{\partial u'}{\partial \hat{z}} \frac{\partial u'}{\partial \hat{z}}) + \epsilon^3 (-\frac{1}{\epsilon R} \frac{\partial^2 u' \hat{u}'}{\partial \hat{x}^2}) = 0 \end{aligned} \quad 4.15$$

$\hat{v}'v'$:

$$\begin{aligned}
& \epsilon^0 (2\hat{v}'v' \frac{\partial \hat{V}}{\partial \hat{y}} + \hat{V} \frac{\partial \hat{v}'v'}{\partial \hat{y}} + \frac{2}{\epsilon R} \frac{\partial \hat{v}'}{\partial \hat{y}} \frac{\partial \hat{v}'}{\partial \hat{y}} + 2\hat{v}' \frac{\partial \hat{p}'}{\partial \hat{y}}) \\
& + \epsilon^{1/2} \frac{\partial \hat{v}'\hat{v}'v'}{\partial \hat{y}} + \epsilon (\frac{\partial \hat{v}'v'}{\partial \hat{t}} + 2\hat{u}'v' \frac{\partial \hat{V}}{\partial \hat{x}} + \hat{U} \frac{\partial \hat{v}'v'}{\partial \hat{x}} - \frac{1}{\epsilon R} \frac{\partial^2 \hat{v}'v'}{\partial \hat{y}^2}) \\
& + \epsilon^{3/2} \frac{\partial \hat{u}'\hat{v}'v'}{\partial \hat{x}} + \epsilon^2 (\frac{2}{\epsilon R} \frac{\partial \hat{v}'}{\partial \hat{x}} \frac{\partial \hat{v}'}{\partial \hat{x}} + \frac{2}{\epsilon R} \frac{\partial \hat{v}'}{\partial \hat{z}} \frac{\partial \hat{v}'}{\partial \hat{z}}) \\
& + \epsilon^3 (-\frac{1}{\epsilon R} \frac{\partial^2 \hat{v}'v'}{\partial \hat{x}^2}) = 0
\end{aligned} \tag{4.16}$$

$\hat{w}'w'$:

$$\begin{aligned}
& \epsilon^0 (\hat{V} \frac{\partial \hat{w}'w'}{\partial \hat{y}} + \frac{2}{\epsilon R} \frac{\partial \hat{w}'}{\partial \hat{y}} \frac{\partial \hat{w}'}{\partial \hat{y}} + 2\hat{w}' \frac{\partial \hat{p}'}{\partial \hat{z}}) + \epsilon^{1/2} \frac{\partial \hat{v}'\hat{w}'w'}{\partial \hat{y}} \\
& + \epsilon^2 (\frac{\partial \hat{w}'w'}{\partial \hat{t}} + \hat{U} \frac{\partial \hat{w}'w'}{\partial \hat{x}}) + \epsilon^{3/2} (\frac{\partial \hat{u}'\hat{w}'w'}{\partial \hat{x}}) \\
& + \epsilon^2 (\frac{2}{\epsilon R} \frac{\partial \hat{w}'}{\partial \hat{x}} \frac{\partial \hat{w}'}{\partial \hat{x}} + \frac{2}{\epsilon R} \frac{\partial \hat{w}'}{\partial \hat{z}} \frac{\partial \hat{w}'}{\partial \hat{z}}) = 0
\end{aligned} \tag{4.17}$$

$\hat{u}'v'$:

$$\begin{aligned}
& \epsilon^0 (\hat{v}'v' \frac{\partial \hat{U}}{\partial \hat{y}} + \hat{u}'v' \frac{\partial \hat{V}}{\partial \hat{y}} + \hat{V} \frac{\partial \hat{u}'v'}{\partial \hat{y}} + \frac{2}{\epsilon R} \frac{\partial \hat{u}'}{\partial \hat{y}} \frac{\partial \hat{v}'}{\partial \hat{y}} + \hat{u}' \frac{\partial \hat{p}'}{\partial \hat{y}} \\
& + \hat{v}' \frac{\partial \hat{p}'}{\partial \hat{x}}) + \epsilon^{1/2} (\frac{\partial \hat{u}'\hat{v}'v'}{\partial \hat{y}}) + \epsilon (\frac{\partial \hat{u}'v'}{\partial \hat{t}} + \hat{u}'v' \frac{\partial \hat{U}}{\partial \hat{x}} \\
& + \hat{u}'\hat{u}' \frac{\partial \hat{V}}{\partial \hat{x}} + \hat{U} \frac{\partial \hat{u}'v'}{\partial \hat{x}} - \frac{1}{\epsilon R} \frac{\partial^2 \hat{u}'v'}{\partial \hat{y}^2}) + \epsilon^{3/2} (\frac{\partial \hat{u}'\hat{u}'v'}{\partial \hat{x}}) \\
& + \epsilon^2 (\frac{2}{\epsilon R} \frac{\partial \hat{u}'}{\partial \hat{x}} \frac{\partial \hat{v}'}{\partial \hat{x}} + \frac{2}{\epsilon R} \frac{\partial \hat{u}'}{\partial \hat{z}} \frac{\partial \hat{v}'}{\partial \hat{z}}) + \epsilon^3 (-\frac{1}{\epsilon R} \frac{\partial^2 \hat{u}'v'}{\partial \hat{x}^2}) = 0
\end{aligned} \tag{4.18}$$

In a manner consistent with the expansion of the Reynolds equations, it is assumed that the flow variables can be

represented as a series expansion of the parameter ϵ .

These expansions take the following form

$$\begin{aligned}
 \hat{U} &= U_0 + \epsilon U_1 + \epsilon^2 U_2 + \dots + \epsilon^m U_m \\
 \hat{V} &= \epsilon V_1 + \epsilon^2 V_2 + \dots + \epsilon^m V_m \quad (\text{since } \hat{V} = 0 \text{ } (\epsilon)) \\
 \hat{u}_i' u_j' &= (u_i' u_j')_0 + \epsilon^{1/2} (u_i' u_j')_1 + \epsilon (u_i' u_j')_2 + \dots \\
 &\quad + \epsilon^{m/2} (u_i' u_j')_m \\
 \hat{u}_i' u_j' u_k' &= (u_i' u_j' u_k')_0 + \epsilon^{1/2} (u_i' u_j' u_k')_1 + \dots \\
 &\quad + \epsilon^{m/2} (u_i' u_j' u_k')_m \\
 \frac{\partial \hat{u}_i}{\partial x_k} \frac{\partial \hat{u}_j}{\partial x_k} &= \left(\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right)_0 + \epsilon^{1/2} \left(\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right)_1 + \dots \\
 &\quad + \epsilon^{m/2} \left(\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right)_m \\
 \hat{u}_i' \frac{\partial p'}{\partial x_j} &= (u_i' \frac{\partial p'}{\partial x_j})_0 + \epsilon^{1/2} (u_i' \frac{\partial p'}{\partial x_j})_1 + \dots \\
 &\quad + \epsilon^{m/2} (u_i' \frac{\partial p'}{\partial x_j})_m
 \end{aligned}$$

4.19

The expansions of the fluctuating quantities of equations 4.19 have been made in terms of $\epsilon^{1/2}$. This was done to

reflect the fact that equations 4.15 through 4.18 were derived by multiplication of the Reynolds equations (which were expanded in terms of ϵ) by a velocity fluctuation u'_1 (which is of order $\epsilon^{1/2}$). This expansion should therefore achieve compatibility, with regard to magnitudes, with the expanded Reynolds equations. Substitution of equations 4.19 into equations 4.15 through 4.18 and collection according to powers of ϵ give the following order of magnitude relationships (for the sake of brevity, only those terms up to and including 0 (ϵ) will be shown. The overmarks indicating nondimensional x and y directions will also be omitted):

$\hat{u'u'}$:

$$\begin{aligned}
 & \epsilon^0 \left[2(u'v')_0 \frac{\partial V_0}{\partial y} + \frac{2}{\epsilon R} \left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} \right)_0 + 2(u' \frac{\partial p'}{\partial x})_0 \right] \\
 & + \epsilon^{1/2} \left[2(u'v')_2 \frac{\partial V_0}{\partial y} + \frac{2}{\epsilon R} \left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} \right)_1 + \frac{\partial (u'u'v')_0}{\partial y} \right. \\
 & + 2(u' \frac{\partial p'}{\partial x})_1 \left. \right] + \epsilon \left[2(u'v')_2 \frac{\partial U_0}{\partial y} + 2(u'v')_0 \frac{\partial U_1}{\partial y} \right. \\
 & + V_1 \frac{\partial (u'u')_0}{\partial y} + \frac{2}{\epsilon R} \left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} \right)_2 + 2(u' \frac{\partial p'}{\partial x})_2 \\
 & + \frac{\partial (u'u'v')_1}{\partial y} + \frac{\partial (u'u')_0}{\partial t} + 2(u'u')_0 \frac{\partial U_0}{\partial x} + U_0 \frac{\partial (u'u')_0}{\partial x} \\
 & \left. - \frac{1}{\epsilon R} \frac{\partial^2 (u'u')_0}{\partial y^2} \right] = 0
 \end{aligned}
 \tag{4.20}$$

$\hat{v}'v'$:

$$\begin{aligned}
& \epsilon^0 \left[\frac{2}{\epsilon R} \left(\frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y} \right)_0 + 2 \left(v' \frac{\partial p'}{\partial y} \right)_0 \right] \\
& + \epsilon^{1/2} \left[\frac{2}{\epsilon R} \left(\frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y} \right)_1 + \frac{\partial (v' v' v')_0}{\partial y} + 2 \left(v' \frac{\partial p'}{\partial y} \right)_1 \right] \\
& + \epsilon \left[2 \left(v' v' \right)_0 \frac{\partial v_1}{\partial y} + v_1 \frac{\partial (v' v')_0}{\partial y} + \frac{2}{\epsilon R} \left(\frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y} \right)_2 \right. \\
& + \frac{\partial (v' v' v')_1}{\partial y} + 2 \left(v' \frac{\partial p'}{\partial y} \right)_2 + \frac{\partial (v' v')_0}{\partial t} + u_0 \frac{\partial (v' v')_0}{\partial x} \\
& \left. - \frac{1}{\epsilon R} \frac{\partial^2 (v' v')_0}{\partial y^2} \right] = 0
\end{aligned} \tag{4.21}$$

$\hat{w}'w'$:

$$\begin{aligned}
& \epsilon^0 \left[\frac{2}{\epsilon R} \left(\frac{\partial w'}{\partial y} \frac{\partial w'}{\partial y} \right)_0 + 2 \left(w' \frac{\partial p'}{\partial z} \right)_0 \right] \\
& \epsilon^{1/2} \left[\frac{2}{\epsilon R} \left(\frac{\partial w'}{\partial y} \frac{\partial w'}{\partial y} \right)_1 + \frac{\partial (v' w' w')_0}{\partial y} + 2 \left(w' \frac{\partial p'}{\partial x} \right)_1 \right] \\
& + \epsilon \left[v_1 \frac{\partial (w' w')_0}{\partial y} + \frac{2}{\epsilon R} \left(\frac{\partial w'}{\partial y} \frac{\partial w'}{\partial y} \right)_2 + 2 \left(w' \frac{\partial p'}{\partial z} \right)_2 \right. \\
& + \frac{\partial (v' w' w')_1}{\partial y} \\
& \left. + \frac{\partial (w' w')_0}{\partial t} + u_0 \frac{\partial w' w'}{\partial x} \right] = 0
\end{aligned} \tag{4.22}$$

$\hat{u}'v'$:

$$\begin{aligned}
& \epsilon^0 \left[(v'v')_0 \frac{\partial U_0}{\partial y} + \frac{2}{\epsilon R} \left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right)_0 + u' \frac{\partial p'}{\partial y}_0 + (v' \frac{\partial p'}{\partial x})_0 \right] \\
& + \epsilon^{1/2} \left[(v'v')_1 \frac{\partial U_0}{\partial y} + \frac{2}{\epsilon R} \left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right)_1 + \frac{\partial (u'v'v')_0}{\partial y} \right. \\
& + \left. (u' \frac{\partial p'}{\partial y})_1 + (v' \frac{\partial p'}{\partial x})_1 \right] \\
& + \epsilon \left[(u'v')_0 \frac{\partial v_1}{\partial y} + (v'v')_2 \frac{\partial U_0}{\partial y} + (v'v')_0 \frac{\partial U_1}{\partial y} + v_1 \frac{\partial (u'v')_0}{\partial y} \right. \\
& + \frac{2}{\epsilon R} \left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right)_2 + \frac{\partial (u'v'v')_1}{\partial y} + (u' \frac{\partial p'}{\partial y})_2 + (v' \frac{\partial p'}{\partial x})_2 \\
& + \frac{\partial (u'v')_0}{\partial t} + (u'v')_0 \frac{\partial U_0}{\partial x} + U_0 \frac{\partial (u'v')_0}{\partial x} \\
& \left. - \frac{1}{\epsilon R} \frac{\partial^2 (u'v')_0}{\partial y^2} \right] = 0
\end{aligned} \tag{4.23}$$

The general validity of the above order of magnitude analysis can be qualitatively determined by comparison with published energy balance plots based on experimental boundary layer data. An assessment of the equations can be made by analyzing the zero order terms of equations 4.20 through 4.23 as they apply to the published test cases for two-dimensional flow ($\bar{U} = f(y)$, $\bar{V} = 0$). For example, it can be seen that production terms only occur in the equations for $\overline{u'u'}$ and $\overline{u'v'}$ and that this production is balanced by a

combination of dissipation and pressure redistribution. All diffusion terms can be seen to be of a lower order of significance. The remaining two equations ($\overline{v'v'}$ and $\overline{w'w'}$) have no production term but instead show that the energy entering the equations through pressure redistribution is balanced by the dissipation term. As in the other equations, the diffusion terms are of less importance. These general observations are verified by the energy balance plots of Townsend (1956) and Laufer (1954) as presented by Hinze (1959, p. 498, p. 531). These comparisons demonstrate that the scaling parameters used were valid in that the final order of magnitude relationships derived in this section are qualitatively confirmed by experimental data.

An evaluation can now be made of the significance of the approximations needed to represent the unknown terms with terms involving Reynolds stresses and mean velocity values. Based on the results of the order of magnitude analysis, it can be seen that both the dissipation terms and the pressure redistribution terms are significant in relation to the production terms. The approximations for these terms must therefore be carefully considered if any effective turbulence model is to result. On the other hand, the triple correlation terms representing diffusion can be approximated to a lower degree of accuracy since these terms have been shown to be less vital in the ultimate solution of the governing equations. The following section provides a

complete description of the selected closure schemes to be used in this model.

SECTION 5

CLOSURE APPROXIMATION SCHEMES

The importance of an adequate and appropriate closure approximation for the dissipation, pressure redistribution, and diffusion terms in the Reynolds transport equations was stated in the previous section. The closure approximation schemes for this turbulence model were selected, or derived, so that difficult flow situations could be properly represented. The following subsections provide a detailed explanation of the various closure schemes used and the reasons for their selection.

Dissipation Closure Approximation

The terms representing dissipation of turbulent energy can be written as shown below

$$2\nu \overline{\left(\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} \right)}$$

The usual closure approach is based on the assumption that turbulence is isotropic and that dissipation is then also isotropic. This leads to the following approximation

$$2\nu \overline{\left(\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} \right)} \approx \frac{2}{3} \nu \delta_{ij} \overline{\left(\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \right)} = \frac{2}{3} \delta_{ij} \epsilon$$

where δ_{ij} is the Kronecker delta, defined as

$$\delta_{ij} = 1 \quad \text{for } i = j$$

$$\delta_{ij} = 0 \quad \text{for } i \neq j$$

and ϵ represents the isotropic dissipation term. This approach is possibly valid for high Reynolds number flows, but may possibly lead to considerable error for reversing flow situations common to tidal estuaries. In view of the importance of the dissipation terms in the governing equations and the belief that dissipation is not isotropic for the specific case of tidally induced flows, a different approach will be presented. The basis of this approach is the two-point velocity correlation. A detailed development of the scheme is presented, first for the auto correlations ($i = j$) and then for the cross correlations ($i \neq j$).

Auto Correlations

The auto correlation portion of the dissipation of turbulent energy term can be written as

$$\overline{\left(\frac{\partial u_i}{\partial x_k} \frac{\partial u_i}{\partial x_k} \right)}$$

Since the following development will be similar for each of the individual terms of the above relationship, consider only the specific term

$$\overline{\left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y}\right)}$$

where the separate derivatives can be expressed as

$$\frac{\partial u'}{\partial y} = \lim_{y \rightarrow y_0} \left[\frac{u'(y) - u'(y_0)}{y - y_0} \right] = \lim_{\delta y \rightarrow 0} \frac{u'_+ - u'}{\delta y}$$

Combination of the two quantities above then yields the following:

$$\overline{\left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y}\right)} = \lim_{\delta y \rightarrow 0} \left(\frac{\overline{u'_+ u'_+} - 2\overline{u'_+ u'} + \overline{u' u'}}{(\delta y)^2} \right)$$

The autocorrelation portion can be defined as

$$\psi = \sqrt{\overline{u'_+ u'_+}}$$

and

$$c = \sqrt{\overline{u' u'}}$$

5.1

By definition, the spacial correlation coefficient f is written as (Schlichting, 1960, p. 467):

$$f(\tau) = \frac{\overline{u'_+ u'}}{\sqrt{\overline{u'_+ u'_+}} \sqrt{\overline{u' u'}}}$$

5.2

where $f(\tau)$ is a function of the separation distance τ . If the definitions of equation 5.1 are used, the dissipation term can be written as

$$\left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} \right) = \lim_{\tau \rightarrow 0} \left(\frac{c^2 - 2f(\tau)uc + c^2}{\tau^2} \right) \quad 5.3$$

where the separation distance τ has been substituted for δy . Two qualities of the correlation coefficient can now be introduced. First, it can be seen that as τ approaches zero $f(\tau)$ approaches unity. This is merely a statement of Cauchy's inequality (Taylor, 1955, p. 204). Secondly, it is assumed that the correlation coefficient $f(\tau)$ is a continuous function and, due to Cauchy's inequality, is a maximum at $\tau = 0$. By definition then, the derivative of $f(\tau)$ with respect to τ , evaluated at the location $\tau = 0$, must be zero. The argument of a continuous function can be made by noting that the quantity $u'_+ u'$ is finite when $\tau = 0$ and $u'_+ \rightarrow u$. In support of this assumption is the often used correlation coefficient relationship

$$f(\tau) = e^{-\frac{\tau^2}{A}}$$

where A is some length parameter of the system. This form of relationship has been used for isotropic, homogeneous turbulence as presented by Batchelor (1956) and Hinze (1959) for a variety of Reynolds numbers. The above qualities can be stated as

$$\lim_{\tau \rightarrow 0} f(\tau) = 1$$

and

$$\lim_{\tau \rightarrow 0} \psi = c$$

If these two relationships are substituted into equation 5.3, it can be seen that both the numerator and denominator approach zero as τ approaches zero. L'Hospital's rule (Taylor, 1955, p. 121) must therefore be used yielding the following:

$$\left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} \right) = \lim_{\tau \rightarrow 0} \left(\frac{2\psi\psi' - 2f'\psi c - 2f\psi'c}{2\tau} \right) \quad 5.4$$

where the primes indicate derivatives with respect to τ and $c \neq c(\tau)$. If the assumptions concerning the function $f(\tau)$ are introduced again, it can be seen that

$$\lim_{\tau \rightarrow 0} \psi' = c'$$

and

$$\lim_{\tau \rightarrow 0} f' = 0$$

Substitution into equation 5.4 still shows that both the numerator and denominator approach zero. Application of L'Hospital's rule is therefore made again resulting in

$$\left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y} \right) = \lim_{\tau \rightarrow 0} \left[\frac{2\psi'\psi' + 2\psi''(\psi - fc) - 4f'\psi'c - 2f''\psi c}{2} \right]$$

since as $\tau \rightarrow 0$, $\psi \rightarrow c$ and $f' \rightarrow 0$, the following simplification can be written

$$\overline{\left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y}\right)} = \frac{2c'c' - 2f''cc}{2}$$

which reduces to the following form

$$\overline{\left(\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y}\right)} = \overline{\left(\frac{\partial}{\partial y} \overline{u'u'}\right)^2} - f'' \overline{u'u'} \quad 5.5$$

This relationship now expresses the dissipation term exactly in terms of the Reynolds stresses. A somewhat similar relationship has been developed for isotropic turbulence (Hinze, 1959, p. 154) which does not include the first term. It is proposed, therefore, that equation 5.5 be used to approximate the autocorrelation dissipation terms in the Reynolds stress transport equations. The approximation of the f'' term will be discussed at a later point.

Cross-correlations

The cross-correlation portion of the dissipation term can be written in the following form

$$\overline{\left(\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}\right)}$$

As in the previous development for the autocorrelations, consider just the following term:

$$\overline{\left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y}\right)}$$

The separate derivatives can be written as

$$\frac{\partial u'}{\partial y} = \lim_{y \rightarrow y_0} \left[\frac{u'(y) - u'(y_0)}{y - y_0} \right] = \lim_{\delta y \rightarrow 0} \frac{u'_+ - u'}{\delta y}$$

and

$$\frac{\partial v'}{\partial y} = \lim_{y \rightarrow y_0} \left[\frac{v'(y) - v'(y_0)}{y - y_0} \right] = \lim_{\delta y \rightarrow 0} \frac{v'_+ - v'}{\delta y}$$

Substitution of the above quantities into the dissipation term results in the following relationship

$$\left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right) = \lim_{\delta y \rightarrow 0} \left[\frac{u'_+ v'_+ - u'_+ v' - u' v'_+ + u' v'}{(\delta y)^2} \right] \quad 5.7$$

A rotation of the axes through an angle α can be accomplished by considering the following coordinate transformation:

$$\begin{aligned} u' &= u^* \cos \alpha - v^* \sin \alpha \\ v' &= u^* \sin \alpha + v^* \cos \alpha \end{aligned} \quad 5.8$$

and let the separation distance $(\delta y)^2$ be represented by τ^2 . If the transformations of equations 5.8 are substituted into equation 5.7 and the angle of rotation α equals 45° , the dissipation term can be written

$$\begin{aligned} \left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right) &= \lim_{\tau \rightarrow 0} \frac{1}{2} (\overline{u_+^* u_+^*} - \overline{v_+^* v_+^*} - \overline{u_+^* u^*} + \overline{v_+^* v^*} - \overline{u^* u_+^*} + \overline{v^* v_+^*} \\ &\quad + \overline{u^* u^*} - \overline{v^* v^*}) \end{aligned} \quad 5.9$$

The above transformation is identical to that shown by Hinze (1959, p. 254). Similar to the autocorrelation development, the following definitions are presented.

$$\begin{aligned}\psi &= \sqrt{\overline{u_+^* u_+^*}} & A &= \sqrt{\overline{u^* u^*}} \\ \delta &= \sqrt{\overline{v_+^* v_+^*}} & B &= \sqrt{\overline{v^* v^*}}\end{aligned}$$

and the two-point velocity correlation coefficients

$$f_{1*}(\tau) = \overline{u_+^* u_+^*} / (\sqrt{\overline{u_+^* u_+^*}} \sqrt{\overline{u^* u^*}})$$

and

$$f_{2*}(\tau) = \overline{v_+^* v_+^*} / (\sqrt{\overline{v_+^* v_+^*}} \sqrt{\overline{v^* v^*}})$$

Equation 5.9 can now be written in the following form by use of the above definitions:

$$\left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right) = \lim_{\tau \rightarrow 0} \frac{1}{2} \left(\frac{\psi^2 - \delta^2 - 2f_{1*}\psi A + 2f_{2*}\delta B + A^2 - B^2}{\tau^2} \right)$$

As previously shown, it can be seen that as $\tau \rightarrow 0$, $\psi \rightarrow A$, $\delta \rightarrow B$, $f_{1*} \rightarrow f_{2*} \rightarrow 1$. Since both the numerator and denominator approach zero as τ approaches zero, L'Hospital's rule is used to give

$$\begin{aligned} & \left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right) \\ &= \lim_{\tau \rightarrow 0} \frac{1}{2} \left(\frac{2\psi\psi' - 2\delta\delta' - 2f_1' \psi A - 2f_1 \psi' A + 2f_2' \delta B + 2f_2 \delta' B}{2\tau} \right) \end{aligned}$$

since $A \neq A(\tau)$ and $B \neq B(\tau)$. L'Hospital's rule is again used since both numerator and denominator approach zero as τ approaches zero. This results in

$$\begin{aligned} \left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right) &= \lim_{\tau \rightarrow 0} \frac{1}{2} (2\psi'\psi' + 2\psi\psi'' - 2\delta'\delta' - 2\delta\delta'' - 2f_1''\psi A \\ &\quad - 4f_1' \psi' A - 2f_1 \psi'' A = 2f_2''\delta B + 4f_2' \delta' B + 2f_2 \delta'' B) / 2 \end{aligned}$$

Since as $\tau \rightarrow 0$; $\psi \rightarrow A$, $\delta \rightarrow B$, $f_1' \rightarrow f_2' \rightarrow 0$, and $f_1 \rightarrow f_2 \rightarrow 1$, the following simplification can be made:

$$\left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right) = \frac{1}{2} (A'A' - B'B' - f_1'' AA + f_2'' BB)$$

which reduces to the following form

$$\left(\frac{\partial u'}{\partial y} \frac{\partial v'}{\partial y} \right) = \frac{1}{2} \left[\left(\frac{\partial \sqrt{u^* u^*}}{\partial \tau} \right)^2 - \left(\frac{\partial \sqrt{v^* v^*}}{\partial \tau} \right)^2 - f_1'' \overline{u^* u^*} + f_2'' \overline{v^* v^*} \right]$$

5.10

The above relationship can be transformed back to the original coordinate system using the following relationships:

$$A) \quad u^* u^* = \frac{u' u' + 2u' v' + v' v'}{2}, \quad v^* v^* = \frac{u' u' - 2u' v' + v' v'}{2}$$

$$B) \quad x = \frac{1}{\sqrt{2}}(x^* - y^*) \quad y = \frac{1}{\sqrt{2}}(x^* + y^*)$$

C) since $\delta\tau = \delta y = \delta\left(\frac{x^* + y^*}{\sqrt{2}}\right)$, it can be seen that

$$\frac{\partial}{\partial\tau} = 2 \frac{\partial}{\partial(x^*+y^*)} = 2 \left[\frac{\partial x}{\partial(x^*+y^*)} \frac{\partial}{\partial x} + \frac{\partial y}{\partial(x^*+y^*)} \frac{\partial}{\partial y} \right]$$

$$\text{or finally that } \frac{\partial}{\partial\tau} = \frac{\partial}{\partial y}$$

$$D) \quad f_{1*} = \frac{1}{\sqrt{2}}(f_1 + f_2) \quad \frac{\partial^2 f_{1*}}{\partial\tau^2} = \frac{1}{\sqrt{2}} \frac{\partial^2 (f_1 + f_2)}{\partial y^2}$$

$$f_{2*} = \frac{1}{\sqrt{2}}(f_2 - f_1) \quad \frac{\partial^2 f_{2*}}{\partial\tau^2} = \frac{1}{\sqrt{2}} \frac{\partial^2 (f_2 - f_1)}{\partial y^2}$$

Substitution of these four relationships into equation 5.10 gives the following

$$\begin{aligned} \left(\frac{\partial \overline{u'}}{\partial y} \frac{\partial \overline{v'}}{\partial y} \right) &= \frac{1}{2} \left[\left(\frac{1}{2} \frac{\partial}{\partial y} (\overline{u'u'} + \overline{\partial u'v'} + \overline{v'v'}) \right)^2 \right. \\ &\quad - \left(\frac{1}{2} \frac{\partial}{\partial y} (\overline{u'u'} - \overline{\partial u'v'} + \overline{v'v'}) \right)^2 \\ &\quad - \frac{1}{2\sqrt{2}} (f_1 + f_2) (\overline{u'u'} + \overline{\partial u'v'} + \overline{v'v'}) \\ &\quad \left. + \frac{1}{2\sqrt{2}} (f_2 - f_1) (\overline{u'u'} - \overline{\partial u'v'} + \overline{v'v'}) \right] \end{aligned} \quad 5.11$$

The following identity will be used to obtain simplified forms of equations 5.5 and 5.11:

$$\left(\frac{\partial \sqrt{\beta}}{\partial y}\right)^2 = \left(\frac{1}{2\sqrt{\beta}} \frac{\partial \beta}{\partial y}\right)^2 = \frac{1}{4\beta} \left(\frac{\partial \beta}{\partial y}\right)^2$$

The final forms for the approximation of the dissipation of turbulence terms can be written as

Autocorrelation:

$$\left(\frac{\partial \overline{u'_i}}{\partial x_k} \frac{\partial \overline{u'_i}}{\partial x_k}\right) = \frac{1}{4\overline{u'_i u'_i}} \left(\frac{\partial \overline{u'_i u'_i}}{\partial x_k}\right)^2 - f''_i \overline{u'_i u'_i}$$

Cross-correlation:

$$\begin{aligned} \left(\frac{\partial \overline{u}}{\partial x_k} \frac{\partial \overline{v}}{\partial x_k}\right) &= \frac{1}{16} \left[\frac{1}{(\overline{u'u'} + \overline{u'v'} + \overline{v'v'})} \left(\frac{\partial \overline{u'u'} + 2\overline{u'v'} + \overline{v'v'}}{\partial k}\right)^2 \right. \\ &\quad \left. - \frac{1}{(\overline{u'u'} - \overline{u'v'} + \overline{v'v'})} \left(\frac{\partial \overline{u'u'} - 2\overline{u'v'} + \overline{v'v'}}{\partial k}\right)^2 \right] \\ &\quad - \frac{\sqrt{2}}{4} [f''_1 (\overline{u'u'} + \overline{v'v'}) + \partial f''_2 \overline{u'v'}] \end{aligned}$$

where repetition of indices is not intended. The above relationships are not isotropic and should therefore represent an improvement over the usual assumption of isotropy of dissipation. This rigorous development was shown because of the relative importance of the dissipation terms in the Reynolds stress transport equations.

Pressure Redistribution Closure Approximation

The terms representing pressure fluctuation effects are as follows

$$\frac{1}{\rho} \left(\overline{p' \frac{\partial u_i'}{\partial x_j}} + \overline{p' \frac{\partial u_j'}{\partial x_i}} \right) - \frac{1}{\rho} \left(\overline{\frac{\partial p'}{\partial x_j} u_i'} + \overline{\frac{\partial p'}{\partial x_i} u_j'} \right) \quad 5.13$$

where the second group of terms represents diffusive transport and the first group represents pressure redistribution. The diffusion portion is generally neglected in relation to the redistribution portion (as previously mentioned, Hanjalic and Launder, 1972) in the modeling of the terms shown in equation 5.13. Results of the order of magnitude analysis represented earlier confirm that other diffusive terms are small; therefore, in this formulation, only the pressure redistribution portion will be considered. This is the accepted practice in most present turbulence models.

An exact expression for the first groups of terms in equation 5.13 was developed by Chou (1945) based on a Green's function formulation of the Poisson equation for pressure fluctuations (presented in Appendix D). Based on Chou's work, as presented by Launder, Reece, and Rodi (1975), this relationship can be written as

$$\frac{p'}{\rho} \frac{\partial u_i'}{\partial x_j} = \frac{1}{4\pi} \int_{vol} \left[\left(\frac{\partial^2 u_j' u_m'}{\partial x_1 \partial x_m} \right) * \left(\frac{\partial u_i'}{\partial x_j} \right) + 2 \left(\frac{\partial u_1'}{\partial x_m} \right) * \left(\frac{\partial u_m'}{\partial x_1} \right) * \left(\frac{\partial u_i'}{\partial x_j} \right) \right] \phi_{ij,1} \phi_{ij,2} \quad 5.14$$

$$\frac{d}{dx} \frac{vol}{|x-x|} + S_{ij}$$

where the terms denoted by an asterisk are evaluated at the fixed vector point x_0 and other terms are evaluated at the moving point $x_0 + x$. The term S_{ij} represents a surface integral which should be negligible away from the vicinity of the wall. The above equation represents two separate processes, the first involving just fluctuating quantities ($\phi_{ij,1}$) and the second involving the mean rate of strain ($\phi_{ij,2}$). Although equation 5.14 is an exact one, it can not be integrated and must be approximated. Unsuccessful attempts to manipulate equation 5.14 led to the conclusion that Rotta's (1951) proposal was the best approximation available. It also represents the most widely used approximation for the pressure redistribution terms.

Rotta's proposal for the $\phi_{ij,2}$ term of equation 5.14 can be written as follows (as reported by Launder, Reece, and Rodi, 1975):

$$\phi_{ij,1} + \phi_{ji,1} = -c_1 (\epsilon/k) (\overline{u_i' u_j'}) - 2/3 \delta_{ij} k \quad 5.15$$

where c_1 is a constant, and k and ϵ are the turbulent kinetic energy and isotropic dissipation rate. According to the above reference, "nearly every worker who has made closure approximations for (equation 5.14) has adopted Rotta's (1951) proposal for $\phi_{ij,1}$." In view of this recommendation, the use of equation 5.15 seems fully justified.

Rotta's proposal for the second term, $\phi_{ij,2}$, was also selected for use in the present model. This term can be

written as follows (as presented by Launder, Reece, and Rodi, 1975):

$$\phi_{ij,2} = \left(\frac{\partial U_1}{\partial x_m} \right) a_{lj}^{mi} \quad 5.16$$

where

$$a_{lj}^{mi} = - \frac{1}{2\pi} \int_{vol} \frac{\overline{\frac{\partial^2 u_m' u_l'}{\partial \xi \partial \xi}}}{|x - x_0|} d vol$$

and ξ 's represent the Cartesian components of the position vector $x - x_0$. According to Rotta, the fourth-order tensor should satisfy the following constraints:

$$\begin{aligned} a_{lj}^{mi} &= a_{lj}^{im} = a_{jl}^{im} \\ a_{li}^{mi} &= 0, \quad a_{jj}^{mi} = \overline{2u_m' u_i'} \end{aligned} \quad 5.17$$

The form of the tensor presented by Launder, Reece, and Rodi (1975) is as follows:

$$\begin{aligned} a_{lj}^{mi} &= \alpha \delta_{lj} \overline{u_m' u_i'} + \beta (\delta_{ml} \overline{u_i' u_j'} + \delta_{mj} \overline{u_i' u_l'} + \delta_{il} \overline{u_m' u_j'} \\ &+ \delta_{ij} \overline{u_m' u_l'}) + c_2 \delta_{mi} \overline{u_l' u_j'} + [\eta \delta_{mi} \delta_{lj} + \nu (\delta_{ml} \delta_{ij} + \delta_{mj} \delta_{il})] k \end{aligned} \quad 5.18$$

where α , β , c_2 , η , and ν are constants. The following relationships result when use is made of the constraints of equation 5.17.

$$\alpha = \frac{1}{11}(4c_2 + 10)$$

$$\beta = -\frac{1}{11}(2 + 3c_2)$$

5.19

$$\eta = -\frac{1}{55}(50c_2 + 4)$$

$$\nu = \frac{1}{55}(20c_2 + 6)$$

Although the above relationships appear to be cumbersome, they are widely used in the field of turbulence modeling and have been found to produce acceptable results.

Diffusion Closure Approximation

The terms representing diffusion by turbulent velocity fluctuations are the following triple velocity correlations

$$\frac{\partial \overline{u'_i u'_j u'_k}}{\partial x_k}$$

An exact transport equation for the triple correlation can be derived through the manipulation of equations C-7 and C-9. The full equation can be written as

$$\begin{aligned} & \frac{\partial \overline{u'_i u'_j u'_k}}{\partial t} + \bar{u}_k \frac{\partial \overline{u'_i u'_j u'_k}}{\partial x_k} + \overline{u'_j u'_l u'_k} \frac{\partial \bar{u}_i}{\partial x_k} + \overline{u'_i u'_l u'_k} \frac{\partial \bar{u}_j}{\partial x_k} + \overline{u'_j u'_i u'_k} \frac{\partial \bar{u}_l}{\partial x_k} \\ & - \overline{u'_j u'_l} \frac{\partial \overline{u'_i u'_k}}{\partial x_k} - \overline{u'_i u'_l} \frac{\partial \overline{u'_j u'_k}}{\partial x_k} - \overline{u'_j u'_i} \frac{\partial \overline{u'_l u'_k}}{\partial x_k} + \frac{\partial \overline{u'_i u'_j u'_k u'_l}}{\partial x_k} \\ & = -\frac{1}{\rho} \overline{u'_j u'_l} \frac{\partial \overline{p'}}{\partial x_i} - \frac{1}{\rho} \overline{u'_j u'_i} \frac{\partial \overline{p'}}{\partial x_l} - \frac{1}{\rho} \overline{u'_i u'_l} \frac{\partial \overline{p'}}{\partial x_j} \end{aligned}$$

$$+ \nu (u'_j u'_l \frac{\partial^2 u'_i}{\partial x_k^2} + u'_j u'_i \frac{\partial^2 u'_l}{\partial x_k^2} + u'_i u'_l \frac{\partial^2 u'_j}{\partial x_k^2}) \quad 5.20$$

As can readily be seen, this equation contains many new unknown correlations. It would be difficult to attempt to scale these terms in a manner similar to those of the previous equations. If it is assumed that the time derivative, convective acceleration, and diffusion terms are of a lower order of significance (as in the Reynolds stress transport equations), some simplification can be achieved. Further simplification can be gained if use is made of the following proposal by Millionshtchikov (1941) as reported by Hanjalic and Launder (1972): "when the triple correlations are small and their distribution properties do not differ substantially from those of a Gaussian one, the quadruple correlations may be approximated in terms of the second-order correlations. . . ." This statement results in the following approximation:

$$\overline{u'_i u'_j u'_k u'_l} = \overline{u'_i u'_j} \cdot \overline{u'_k u'_l} + \overline{u'_i u'_k} \cdot \overline{u'_j u'_l} + \overline{u'_i u'_l} \cdot \overline{u'_k u'_j} \quad 5.21$$

Use of equation 5.21 reduces the sum of the quadruple correlations and the terms involving Reynolds stresses and derivatives of Reynolds stresses to

$$- (\overline{u'_i u'_l} \frac{\partial \overline{u'_j u'_k}}{\partial x_1} + \overline{u'_j u'_l} \frac{\partial \overline{u'_k u'_i}}{\partial x_1} + \overline{u'_k u'_l} \frac{\partial \overline{u'_i u'_j}}{\partial x_1}) \quad 5.22$$

The above simplifications leave only the terms involving pressure fluctuations to be accounted for. If an analogy exists between the Reynolds stress transport equations and the equations of 5.20, it cannot be assumed that the pressure fluctuation terms are insignificant. In view of the above, and the desire to use an accepted closure scheme, it was decided to represent the pressure fluctuation terms by the same approximation presented by Hanjalic and Launder (1972). Their assumption of proportionality is

$$- (\overline{u_i' u_j' \frac{\partial p'}{\partial x_k}} + \overline{u_j' u_k' \frac{\partial p'}{\partial x_i}} + \overline{u_k' u_i' \frac{\partial p'}{\partial x_j}}) \propto - \frac{\epsilon}{k} \overline{u_i' u_j' u_k'} \quad 5.23$$

The use of equations 5.22 and 5.23 gives the following approximation for the triple correlation terms representing diffusion:

$$\overline{u_i' u_j' u_k'} = - c_s \frac{k}{\epsilon} (\overline{u_i' u_l'} \frac{\partial \overline{u_j' u_k'}}{\partial x_l} + \overline{u_j' u_l'} \frac{\partial \overline{u_k' u_i'}}{\partial x_l} + \overline{u_k' u_l'} \frac{\partial \overline{u_i' u_j'}}{\partial x_l}) \quad 5.24$$

where c_s is a constant. Although severe simplifications were used in the derivation of equation 5.24, it is assumed justified in view of the order of magnitude analysis showing the triple correlations to be of less significance than other terms in the Reynolds stress transport equations.

The closure approximations presented in subsections 5.1, 5.2, and 5.3 along with the Reynolds equations and continuity equation now represent a closed set of equations for

turbulent flow. This second order closure model was developed with a minimum of simplifying assumptions so that the final set of governing equations would be as general as possible. This requirement was felt to be essential since the ultimate application of the model will be to tidally induce flow regimes.

SECTION 6

THE TURBULENCE MODEL

A consolidation of the previous sections can now be made so that a completely closed set of equations for turbulent flow of a shallow-water wave can be realized. To accomplish this task, the Reynolds equations, continuity equation, and Reynolds stress transport equations must be redimensionalized and recombined to an appropriate and equivalent order of magnitude. The level of accuracy selected for this model was based on the Reynolds stress transport equations such that the lowest order of ϵ relationship containing diffusion terms was included. This selection was made so that the final set of equations would be parabolic in form. Similar reasoning was made by Launder, Reece, and Rodi (1975) in their formulation of two-dimensional shear flows. Inspection of equations 4.20 through 4.23 shows this level of approximation to be $O(\epsilon^{1/2})$. The equivalent order of accuracy for the Reynolds equations and the continuity equation is $O(\epsilon)$. The terms including these orders of magnitude will therefore represent the basic turbulence model.

In order to explicitly account for a variable water-surface boundary, use is made of the continuity equation

and the boundary condition equations (presented in Appendix B) to express the Reynolds equations in a form containing the water-surface elevation. This is done by first recombining the Reynolds equations and continuity equation to an $O(\epsilon)$ approximation as shown below:

$$\frac{\partial \bar{U}}{\partial t} + \bar{U} \frac{\partial \bar{U}}{\partial x} + \bar{V} \frac{\partial \bar{U}}{\partial y} = - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \nu \frac{\partial^2 \bar{U}}{\partial x^2} - \frac{\partial \overline{u'v'}}{\partial y} \quad 6.1$$

$$\frac{1}{\rho} \frac{\partial \bar{P}}{\partial y} + g + \frac{\partial \overline{v'v'}}{\partial y} = 0 \quad 6.2$$

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0 \quad 6.3$$

Equations 6.1 through 6.3 can now be used to develop the governing equations in terms of the water-surface elevation instead of the pressure term. The development of this begins with the integration of the continuity equation (6.3) with respect to y as shown

$$\int_{-H}^{\zeta} \frac{\partial \bar{U}}{\partial x} dy + \int_{-H}^{\zeta} \frac{\partial \bar{V}}{\partial y} dy = 0$$

where ζ is the water-surface elevation and $-H$ is the bottom. Integration can be carried out by application of Leibnitz's rule for interchanging the order of differentiating and integrating (Taylor, 1955, p. 523). This yields the following:

$$\frac{\partial}{\partial x} \int_{-H}^{\zeta} \bar{U} dy + (\bar{V}_S - \bar{U}_S \frac{\partial \zeta}{\partial x}) - (\bar{V}_B + \bar{U}_B \frac{\partial H}{\partial x}) = 0$$

where the subscripts represent the mean velocity values at the surface (s) and bottom (B). Use of the boundary condition equations B.1 and B.2 reduce the above equation to one involving the water-surface elevation, as shown

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} \int_{-H}^{\zeta} \bar{U} dy$$

Approximations for the integral portion can be made by various methods, such as the trapezoidal rule. Equation 6.2 can also be integrated with respect to y, for a constant density fluid, as follows:

$$\frac{1}{\rho} \int_{-h}^{\zeta} \frac{\partial \bar{P}}{\partial y} dy + g \int_{-h}^{\zeta} dy + \int_{-h}^{\zeta} \frac{\partial \overline{v'v'}}{\partial y} dy = 0$$

resulting in

$$\frac{1}{\rho} [-\bar{P}(h)] + g(\zeta + h) + [\overline{v'v'}(\zeta) - \overline{v'v'}(-h)] = 0$$

where -h is an arbitrary distance below the mean water level. Since it has been shown that both $\frac{\partial \overline{v'v'}}{\partial x}(\zeta)$ and $\frac{\partial \overline{v'v'}}{\partial x}(-h)$ are $O(\epsilon^2)$, they are neglected. This relationship can be substituted into equation 6.1 for the pressure term. The final forms of the Reynolds equations and the continuity equation

to be used in the model can be written in the following form:

$$\frac{\partial \bar{U}}{\partial t} + \bar{U} \frac{\partial \bar{U}}{\partial x} + \bar{V} \frac{\partial \bar{U}}{\partial y} = -g \frac{\partial \bar{\zeta}}{\partial x} + \nu \frac{\partial^2 \bar{U}}{\partial y^2} - \frac{\partial \overline{u'v'}}{\partial y} \quad 6.4$$

$$\frac{\partial \bar{\zeta}}{\partial t} + \frac{\partial}{\partial x} \int_{-H}^{\zeta} \bar{U} dy = 0 \quad 6.5$$

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0 \quad 6.6$$

Recombination of the Reynolds stress transport equations to $O(\epsilon^{1/2})$ results in the following governing equations:

$$2\overline{u'v'} \frac{\partial \bar{U}}{\partial y} + 2\nu \frac{\partial \overline{u'}}{\partial y} \frac{\partial \overline{u'}}{\partial y} - \frac{2}{\rho} \overline{p'} \frac{\partial \overline{u'}}{\partial x} + \frac{\partial \overline{u'u'v'}}{\partial y} = 0 \quad 6.7$$

$$2\nu \frac{\partial \overline{v'}}{\partial y} \frac{\partial \overline{v'}}{\partial y} - \frac{2}{\rho} \overline{p'} \frac{\partial \overline{v'}}{\partial y} + \frac{\partial \overline{v'v'v'}}{\partial y} = 0 \quad 6.8$$

$$2\nu \frac{\partial \overline{w'}}{\partial y} \frac{\partial \overline{w'}}{\partial y} - \frac{2}{\rho} \overline{p'} \frac{\partial \overline{w'}}{\partial z} + \frac{\partial \overline{v'w'w'}}{\partial y} = 0 \quad 6.9$$

$$\overline{v'v'} \frac{\partial \bar{U}}{\partial y} - \frac{\overline{p'}}{\rho} \left(\frac{\partial \overline{u'}}{\partial y} \frac{\partial \overline{v'}}{\partial x} \right) + 2\nu \frac{\partial \overline{u'}}{\partial y} \frac{\partial \overline{v'}}{\partial y} + \frac{\partial \overline{u'v'v'}}{\partial y} = 0 \quad 6.10$$

The applicable closure approximations, as derived in Section 5, are as follows:

$$\frac{\partial \overline{u'u'v'}}{\partial y} = \frac{\partial}{\partial y} \left[-c_s \frac{k}{\varepsilon} \left(2\overline{u'u'} \frac{\partial \overline{u'v'}}{\partial x} + 2\overline{u'v'} \frac{\partial \overline{u'v'}}{\partial y} + \overline{u'v'} \frac{\partial \overline{u'u'}}{\partial x} + \overline{v'v'} \frac{\partial \overline{u'u'}}{\partial y} \right) \right] \quad 6.11$$

$$\frac{\partial \overline{v'v'v'}}{\partial y} = \frac{\partial}{\partial y} \left[-c_s \frac{k}{\varepsilon} \left(3\overline{u'v'} \frac{\partial \overline{v'v'}}{\partial x} + 3\overline{v'v'} \frac{\partial \overline{v'v'}}{\partial y} \right) \right] \quad 6.12$$

$$\frac{\partial \overline{v'w'w'}}{\partial y} = \frac{\partial}{\partial y} \left[-c_s \frac{k}{\varepsilon} \left(\overline{u'v'} \frac{\partial \overline{w'w'}}{\partial x} + \overline{v'v'} \frac{\partial \overline{w'w'}}{\partial y} \right) \right] \quad 6.13$$

$$\frac{\partial \overline{u'v'v'}}{\partial y} = \frac{\partial}{\partial y} \left[-c_s \frac{k}{\varepsilon} \left(2\overline{u'v'} \frac{\partial \overline{u'v'}}{\partial x} + 2\overline{v'v'} \frac{\partial \overline{u'v'}}{\partial y} + \overline{u'u'} \frac{\partial \overline{v'v'}}{\partial x} + \overline{u'v'} \frac{\partial \overline{v'v'}}{\partial y} \right) \right] \quad 6.14$$

$$\left(\frac{\partial \overline{u'}}{\partial y} \right)^2 = \frac{1}{4\overline{u'u'}} \left(\frac{\partial \overline{u'u'}}{\partial y} \right)^2 - f''_u(\overline{u'u'}) \quad 6.15$$

$$\left(\frac{\partial \overline{v'}}{\partial y} \right)^2 = \frac{1}{4\overline{v'v'}} \left(\frac{\partial \overline{v'v'}}{\partial y} \right)^2 - f''_v(\overline{v'v'}) \quad 6.16$$

$$\left(\frac{\partial \overline{w'}}{\partial y} \right)^2 = \frac{1}{4\overline{w'w'}} \left(\frac{\partial \overline{w'w'}}{\partial y} \right)^2 - f''_w(\overline{w'w'}) \quad 6.17$$

$$\begin{aligned} \left(\frac{\partial \overline{u'}}{\partial y} \frac{\partial \overline{v'}}{\partial y} \right) &= \frac{1}{16} \left[\frac{1}{(\overline{u'u'} + \partial \overline{u'v'} + \overline{v'v'})} \left(\frac{\partial (\overline{u'u'} + \partial \overline{u'v'} + \overline{v'v'})}{\partial y} \right)^2 \right. \\ &\quad - \frac{1}{(\overline{u'u'} - \partial \overline{u'v'} + \overline{v'v'})} \left(\frac{\partial (\overline{u'u'} + \partial \overline{u'v'} + \overline{v'v'})}{\partial y} \right)^2 \\ &\quad \left. - \frac{\sqrt{2}}{4} [f''_u(\overline{u'u'} + \overline{v'v'}) + 2f''_v(u'v')] \right] \quad 6.18 \end{aligned}$$

The pressure fluctuation approximation terms can be simplified by consideration of orders of magnitude of the expanded terms. For instance, it has been shown that

$$\overline{u'_i u'_j} \frac{\partial \bar{U}}{\partial x} = O(\epsilon^2)$$

$$\overline{u'_i u'_j} \frac{\partial \bar{V}}{\partial x} = O(\epsilon^3)$$

$$\overline{u'_i u'_j} \frac{\partial \bar{V}}{\partial y} = O(\epsilon^2)$$

$$\overline{u'_i u'_j} \frac{\partial \bar{U}}{\partial y} = O(\epsilon)$$

By use of these results, the pressure fluctuation approximations can be written as follows:

$$\overline{p' \frac{\partial u'}{\partial x}} = -\frac{c_1}{2} \frac{\epsilon}{k} (\overline{u' u'} - 2/3k) + \frac{\partial \bar{u}}{\partial y} \left(\frac{2}{11} \overline{u' v'} \right) (3 - c_2) \quad 6.19$$

$$\overline{p \frac{\partial v'}{\partial y}} = -\frac{c_1}{2} \frac{\epsilon}{k} (\overline{v' v'} - 2/3k) + \frac{\partial \bar{u}}{\partial y} \left(\frac{5c_2 - 4}{11} \right) \overline{u' v'} \quad 6.20$$

$$\overline{p \frac{\partial w'}{\partial z}} = -\frac{c_1}{2} \frac{\epsilon}{k} (\overline{w' w'} - 2/3k) + \frac{\partial \bar{u}}{\partial y} (\overline{u' v'}) \left(\frac{-2 + 3c_2}{11} \right) \quad 6.21$$

$$\begin{aligned} \overline{p \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right)} &= c_1^* \frac{\epsilon}{k} (\overline{u' v'}) + \frac{\partial \bar{u}}{\partial y} \left[\left(\frac{25c_2 - 9}{55} \right) \overline{u' u'} \right. \\ &\quad \left. + \overline{v' v'} \left(\frac{41 - 20c_2}{55} \right) + \overline{w' w'} \left(\frac{1 - 15c_2}{55} \right) \right] \quad 6.22 \end{aligned}$$

where the values of the various coefficients will be discussed at a later point.

Although this system of equations appears to be quite formidable, it does represent a closed system of equations. Through the judicious selection of an appropriate numerical solution scheme and an appropriate sequence of solution, equations 6.4-6.10 can be solved for the desired unknown quantities \bar{u} , \bar{v} , $\bar{\zeta}$, $\overline{u'u'}$, $\overline{v'v'}$, $\overline{w'w'}$, and $\overline{u'v'}$. This procedure will be described in the following section in addition to the applicable boundary and initial conditions.

SECTION 7

THE NUMERICAL APPROXIMATION

In this section, attention is turned to the numerical procedure and associated boundary and initial conditions necessary to arrive at a numerically stable approximation of the governing equations. This must begin with one of the most basic parameters of any numerical model, the vertical grid spacing. An equally spaced vertical grid was selected as one-tenth of the depth of flow. The rationale of this selection is as follows: inspection of the turbulence data of Reichardt (1938) and Klebanoff (1955), as presented in Schlichting (1960, p. 466, 467), and the data of Laufer (1954), as presented in Hinze (1959, p. 521) show that a distance of one-tenth of the flow dimension from the wall is outside the viscous sublayer in which the Reynolds stress values increase sharply from zero to some maximum value. By selecting a grid spacing of one-tenth the flow depth, the difficulties of predicting this inflection point are eliminated. This selection not only eases potential computational difficulties involving the approximation of the various terms but also keeps the computer memory requirements to a minimum. Memory requirements and associated

computer run times are an important consideration if the final model is ever to be used in a practical sense.

The numerical procedure employed in the model assumes an initial mean velocity profile and solves for the Reynolds stresses for that profile through an explicit iteration scheme. The mechanics of actually doing this involve making initial estimates of the Reynolds stresses and depth-dependent coefficients and updating those estimates during each iteration. The resulting Reynolds stress profile at each x location can then be used to compute a new mean velocity profile and water-surface profile for the entire length of flow at the new time-step $t + \Delta t$. A detailed step-by-step description of this procedure will be made following a description of A) the initial and subsequent estimates of depth-dependent coefficients and B) the boundary conditions.

A) Depth-Dependent Coefficients. Estimates of the depth-dependent parameters of equations 6.11-6.22 must be made before and during the iteration procedure. These estimates are described below.

1. ϵ/k . The value of ϵ/k (or inversely k/ϵ) found in equations 6.11-6.14 and 6.19-6.22 is computed from the $0(\epsilon^0)$ formulation of equations 6.7-6.10 (neglecting the diffusion terms) as follows. By definition, the isotropic dissipation ϵ is expressed as

$$2\nu \left(\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right) = \frac{2}{3} \epsilon_{ij} \epsilon$$

or

$$\epsilon = \nu \left(\frac{\partial u'}{\partial x_k} \frac{\partial u'}{\partial x_k} + \frac{\partial v'}{\partial x_k} \frac{\partial v'}{\partial x_k} + \frac{\partial w'}{\partial x_k} \frac{\partial w'}{\partial x_k} \right)$$

Since only derivatives with respect to y are considered and the pressure redistribution terms vanish due to continuity, the sum of equations 6.7-6.10 yields the following relationship

$$\overline{u'v'} \frac{\partial \bar{U}}{\partial y} + \epsilon = 0 \quad 7.1$$

Division by the quantity $k = 1/2(\overline{u'u'})$ then gives the value of ϵ/k for each time-step and for each grid point as shown

$$\epsilon/k = 2 \overline{u'v'} \frac{\partial \bar{U}}{\partial y} / (\overline{u'u'} + \overline{v'v'} + \overline{w'w'}) \quad 7.2$$

This coefficient is initially calculated from the input Reynolds stress and velocity profile data and subsequently computed from the values generated by the model. This procedure avoids solving transport equations for ϵ and k as is done by Hanjalic and Launder (1972) and Launder, Reece, and Rodi (1975).

2. Two-point velocity correlation coefficients. The two-point velocity correlation coefficient derivatives found in the dissipation approximations of equations 6.15-6.18 are also first approximated from the $O(\epsilon^0)$ formulation of equations 6.7-6.10. These first estimates are merely calculated by balancing the sum of the production and/or pressure redistribution terms with the diffusion term. This procedure was used by Launder, Reece, and Rodi (1975) as evidenced by the following excerpt: "the initial value of the dissipation rate was deduced as the closing term in a turbulence energy balance." Subsequent updatings of the derivative of the velocity correlation coefficient are based on the common supposition that the dissipation is proportional to the turbulent kinetic energy as shown by the following expression (Rodi, 1980, p. 21)

$$\epsilon = \frac{C_D k^{3/2}}{L}$$

It is therefore assumed that each individual dissipation term should be updated to reflect an overall increase or decrease proportional to a 3/2 power of the respective Reynolds stress. This is achieved by updating the correlation coefficient at the end of each iteration, based on the new value of the Reynolds stress calculated during that iteration, using the following relationship:

$$f_i''(\tau + \Delta\tau) = f_i''(\tau) \sqrt{\frac{\overline{u_i' u_i'}(\tau + \Delta\tau)}{\overline{u_i' u_i'}(\tau)}}$$

where no summation of indices is intended. This procedure was found to be necessary and effective in the solution of the Reynolds stress transport equations.

3. C_1^* . The value of C_1^* in equation 6.22 was intended to be the constant C_1 in the Launder, Reece, and Rodi (1975) formulation. In order to account for wall effects, they proposed an additional near-wall correction term. Instead of using their formulation, the following simpler correction term was used:

$$C_1^* = (C_3 + C_3/1)$$

where 1 is the grid point number measured from the wall, i.e., $1 = 1, 2, 3, \dots, 10$. This was found to be quite effective.

4. k/ϵ . The value of k/ϵ used in equations 6.11-6.13 was merely calculated as the reciprocal of ϵ/k described in section a) with the following exception: because of the $O(\epsilon)$ formulation of ϵ/k used in equation 7.1, it can be seen that when either $\overline{u'v'}$ or the velocity gradient approach zero, the value of ϵ will approach zero. The reciprocal then approaches infinity. This possibility sometimes

occurs near the free surface during the computation. Since the term k/ϵ only occurs in the diffusion term approximations and it is recognized that these are small terms, especially near the free surface, the following correction was used:

$$C_s = C_s(\epsilon/k) \quad \text{when } \epsilon/k < .40$$

and

$$\epsilon/k = .20 \quad \text{when } \epsilon/k < .20$$

These corrections have no theoretical basis and are only used to force the diffusion terms near zero close to the free surface where the singularity may occur. This is consistent with energy balance plots showing minimal diffusion at the surface (Hinze, 1959).

B) Boundary Conditions. Boundary conditions used for the governing equations can be placed in the following five categories: (1) upstream, (2) downstream, (3) surface, (4) wall, and (5) other. These are addressed separately as follows.

1. Upstream. The upstream boundary condition of known discharge was maintained; however, the water level and velocity profile were allowed to vary based on the

following mass conservation considerations. Equation 6.5 can be written as

$$\frac{\partial \zeta}{\partial t} = - \frac{\partial q}{\partial x} \quad 7.3$$

where q is mass flux at a given point per unit mass and width of channel. This value is approximated by the trapezoidal rule as follows (Hornbeck, 1975)

$$q = \int_a^b \bar{U} dy = \frac{\Delta y}{2} [\bar{U}(a) + \bar{U}(b) + 2 \sum_{j=1}^{m-1} \bar{U}(j)]$$

where $\bar{U}(a) = 0$ = velocity at the wall and $\bar{U}(b)$ = surface velocity. Equation 7.3 can be written in finite difference form as follows:

$$\frac{\zeta_1' - \zeta_1^0}{\Delta t} = - \frac{(q_2 - q_1)}{\Delta x}$$

Averaging in space and time then gives the following

$$(\zeta_1' + \zeta_2' - \zeta_1^0 - \zeta_2^0) \Delta x = -(q_2' + q_2^0 - q_1' - q_1^0) \Delta t \quad 7.4$$

where superscripts refer to the new and old time-steps and the subscripts refer to the upstream boundary and the second grid location. The new water level can then be calculated as follows:

$$\zeta_1' = \zeta_1^0 - \zeta_2' + \zeta_2^0 + (q_1' + q_1^0 - q_2' - q_2^0)\Delta t/\Delta x \quad 7.5$$

A predictor-corrector based new water level was then defined as

$$\zeta_1' = 1/2(\zeta_1' + \zeta_1^0) \quad 7.6$$

Assuming constant inflow discharge, the velocity distribution can be adjusted based on the change in water level as follows

$$(\bar{u}_1')_j = (\bar{u}_1^0)_j \zeta_1^0/\zeta_1' \quad 7.7$$

where the subscript j refers to the vertical grid points.

2. Downstream. The downstream boundary condition of known water-level elevation was maintained. The velocity distribution and associated discharge were allowed to vary according to an argument similar to that used for the upstream boundary. Equation 7.4 can be written for the flux at the $N + 1$ (downstream) boundary as follows:

$$q_{N+1}' = -(\zeta_N' + \zeta_{N+1}^0 - \zeta_N^0 - \zeta_{N+1}^0)\Delta x/\Delta t + q_N' + q_N^0 - q_{N+1}^0$$

yielding the new time-step flux at the downstream boundary. The corresponding velocity profile was then calculated using a similar profile assumption as follows:

$$(u'_{N+1})_j = (u^o_{N+1})_j q'_{N+1}/q^o_{N+1} \quad 7.8$$

where the subscript j refers to the vertical grid points.

3. Wall. A no-slip boundary condition for the mean velocity components and individual Reynolds stresses was imposed at the wall. In order to calculate the derivative with respect to y of the Reynolds stresses for the first point from the wall, however, a "wall" value was calculated for each Reynolds stress. These values were then used only for a central difference approximation of the derivatives at the point adjacent to the wall. This "wall" value for the normal stresses $\overline{u'u'}$, $\overline{v'v'}$, and $\overline{w'w'}$ was calculated from a linear back extrapolation of the respective Reynolds stress values at the first and second grid point from the wall as calculated from the previous time-step. This procedure appears consistent with published data since the first point is outside the viscous sublayer. A further inspection of this data shows that the $\overline{u'v'}$ curve is relatively flat at the tenth point, in contrast with the normal Reynolds stress terms which exhibit a definite slope. For this reason, the "wall" value for $\overline{u'v'}$ was set equal to the value calculated for the first point from the wall during the previous time-step. The derivative of $\overline{u'v'}$ was then calculated by a central difference approximation using the "wall" value and the values at the first and second point from the wall.

This condition also appears reasonable and provides acceptable results.

4. Surface. The surface boundary condition imposed was that of symmetry of the mean velocity profile. This condition forces the first and second derivative of \bar{U} with respect to y to zero at the surface. An additional boundary condition of $\overline{u'v'} = 0$ was imposed at the surface. The reasoning behind this is as follows: the value of $\overline{u'v'}$ is always less than zero when the mean velocity gradient is positive ($\frac{\partial \bar{u}}{\partial y} > 0$). Conversely, it is greater than zero when the velocity gradient is negative. This is a consequence of production being balanced partially by dissipation, which is a positive-definite quantity. This is verified by Hinze (1959, p. 252). Since the sign of $\overline{u'v'}$ is opposite that of the velocity gradient, and the condition of symmetry of the mean velocity profile has been imposed, the condition of $\overline{u'v'} = 0$ at the surface is reasonable. Experimental data also show this to be a reasonable assumption. Normal Reynolds stress values for the surface were computed by extrapolation assuming a zero derivative with respect to y at the surface. This also appears to be a reasonable assumption and eliminates the necessity of fixing the Reynolds stress values at the surface. In addition to this approximation, the parameter ϵ/k is set equal to the value calculated for the point adjacent to the surface. This

is of negligible consequence since the value at that point is essentially zero.

5. Other. One additional boundary condition must be imposed on the Reynolds stresses. The necessity for this can be seen by inspection of the Reynolds stress transport equations 6.7-6.10, the associated approximations 6.11-6.22, and the imposed surface and bed boundary conditions of $\bar{U} = \bar{V} = \overline{u'u'} = \overline{v'v'} = \overline{w'w'} = 0$ at the bed and $\partial\bar{U}/\partial y = \partial^2\bar{U}/\partial y^2 = \overline{u'v'} = 0$ at the surface. The resulting equations then constitute a linear homogeneous set of simultaneous equations, the solution of which is either the trivial solution or an eigenvalue solution. In order to arrive at a proper solution, a nonzero boundary condition must be imposed. This is determined from the Reynolds equation (6.4) by first writing the equation for the surface as follows:

$$\frac{\partial\bar{U}}{\partial t} + \bar{U} \frac{\partial\bar{U}}{\partial x} + \bar{V} \frac{\partial\bar{U}}{\partial y} = -g \frac{\partial\bar{\zeta}}{\partial x} - \frac{\partial\overline{u'v'}}{\partial y}$$

If the condition of steady uniform flow, which is the condition for which this model will be tested, is considered, the resulting equation becomes

$$\frac{\partial\overline{u'v'}}{\partial y} = -g \frac{\partial\bar{\zeta}}{\partial x} = \text{constant} \quad 7.9$$

for a given water-surface slope corresponding to normal flow. Since the value of $\overline{u'v'}$ is zero at the surface, the additional boundary condition is calculated, in backward difference form, as follows

$$\overline{u'v'}_{K-1} = gS\Delta y \quad 7.10$$

where S is the water-surface slope and the subscript $k-1$ refers to the first grid point below the surface.

Before a detailed sequence-of-events description of the model is presented, the iteration scheme used to approximate the Reynolds stresses is described. Each Reynolds stress equation (equations 6.7-6.10) is first expressed as the sum of its respective production, dissipation, pressure redistribution, and diffusion segments. These resulting equations are then written in the form of a heat equation as expressed below

$$\begin{aligned} \frac{\partial u'_i u'_j}{\partial \tau} &= -[\text{PROD}(u'_i u'_j) + \text{DISS}(u'_i u'_j) + \text{REDIS}(u'_i u'_j) + \text{DIFF}(u'_i u'_j)] \\ &= \text{SUM}(u'_i u'_j) = 0 \end{aligned} \quad 7.11$$

where τ is some small parameter. A single computational loop is then written in finite difference form containing the following functional relationships:

$$\overline{u'u'}(\tau + \Delta\tau) = \overline{u'u'}(\tau) - \Delta\tau(\text{SUM}uu)$$

$$\overline{v'v'}(\tau + \Delta\tau) = \overline{v'v'}(\tau) - \Delta\tau(\text{SUM}vv)$$

7.12

$$\overline{w'w'}(\tau + \Delta\tau) = \overline{w'w'}(\tau) - \Delta\tau(\text{SUM}ww)$$

$$\overline{u'v'}(\tau + \Delta\tau) = \overline{u'v'}(\tau) - \Delta\tau(\text{SUM}uv)$$

The iteration procedure then repeats this loop computing a new Reynolds stress value at the new τ value at the end of each iteration. This procedure minimizes the sum terms thereby converging on a stable Reynolds stress value for the given boundary conditions.

The following represents a step-by-step description of the sequence of events necessary for one completed time iteration of the numerical model. It is applied to a section of channel flow which is represented by $I_{\max} + 1$ grid points in the horizontal direction and $K_{\max} + 1$ grid points in the vertical. The grid points corresponding to $I = 1$ represent the upstream boundary, while those corresponding to $I = I_{\max} + 1$ represent the downstream boundary. Grid points on the bottom, or wall, are represented by $K = 1$, while the surface is represented by $K = K_{\max} + 1$. This represents I_{\max} by K_{\max} grid cells. The final step will then represent the solution for the entire flow field at the time $t = t_0 + \Delta t$.

Sequence of Events:

1. The initial values for the flow field (\bar{U}_i , $\bar{\epsilon}$ and $\overline{u_i' u_j'}$) are set equal to the experimental values.
2. The initial values for the two-point velocity correlation coefficient derivatives are calculated.
3. The time loop begins for the calculation of the flow field for the time $t = t_0 + \Delta t$ where t_0 represents some arbitrary starting time.
4. Mass flux at the upstream boundary is calculated for $t = t_0$.
5. The calculation loop begins for the horizontal grid points $I = 2, I_{\max} + 1$.
6. Initial values of the parameter ϵ/k are computed for each vertical grid point $K = 1, K_{\max} + 1$ at the I grid section.
7. The iteration loop begins for the calculation of the Reynolds stresses at the I grid section.
For the dummy variable τ ,
 - a) "wall" Reynolds stresses are calculated for the I grid section corresponding to $K = 1$ for τ ,

- b) Reynolds stresses are calculated for the remaining grid points $K = 2, K_{\max}$ for τ ,
 - c) two-point velocity correlation coefficient derivatives are updated for each grid point $K = 2, K_{\max}$ based on the Reynolds stress values at τ ,
 - d) the parameter ϵ/k is recalculated using the newly calculated values of Reynolds stresses and the initial mean velocity gradient for each grid point $K = 2, K_{\max}$ at τ ,
 - e) the normal Reynolds stresses and the parameter ϵ/k are defined for the surface as described under boundary conditions,
 - f) repeat step 7 for the new iteration step $\tau = \tau + \Delta\tau$.
8. Calculate the mean horizontal velocity component for $t = t_0 + \Delta t$ for the grid points $K = 2, K_{\max}$ and $I = 2, I_{\max}$ using central difference approximations.
 9. Calculate the surface mean horizontal velocity component for $t = t_0 + \Delta t$ at the grid point $K = K_{\max} + 1$ using backward difference approximation for the grid sections $I = 2, I_{\max}$.

10. Compute the water-surface elevation for the grid sections $I = 2, I_{\max}$ for $t = t_0 + \Delta t$.
11. Compute the mean velocity profile at the downstream boundary for $t = t_0 + \Delta t$.
12. Compute the water-surface elevation for the upstream boundary for $t = t_0 + \Delta t$.
13. Compute the mean velocity profile at the upstream boundary for $t = t_0 + \Delta t$.
14. Return to step 3 to begin calculations for the time-step $t = t_0 + 2\Delta t$.

These fourteen steps briefly describe the computational procedure involved in the model. The next section will describe the actual application of the model to a set of experimental data obtained in a laboratory flume.

SECTION 8

VERIFICATION OF THE MODEL

Verification of the model to experimentally measured data was performed to demonstrate that the concepts used in the derivation of the governing equations are valid. The data selected for this comparison are those presented by McQuivey (1973). Test facilities included a 20-centimeter wide flume with smooth rigid boundaries, both a cylindrical-shaped and a wedge-shaped hot-film sensor, and a calibrated 1/8-inch-diameter pitot tube. Further details of the equipment and measuring techniques are presented by McQuivey. Mean flow parameters and variables for the particular test are shown in Tables 1 and 2.

Table 1
Mean Flow Parameters

S = Slope of the Energy Gradient	= 1.75×10^{-3}
W = Width of Flume	= 0.632 ft.
Q = Discharge	= 0.0874 cfs
T = Temperature	= 27.2°C
D = Depth of Flow (Channel Center Line)	= 0.1013 ft.
\bar{V} = Average Velocity (Q/A)	= 1.36 ft/sec
R = Reynolds Number ($\bar{V}D/\nu$)	= 15,000
F = Froude Number ($\bar{V}\sqrt{gD}$)	= 0.75
C/ g = Chezy Discharge Coefficient	= 20.7
ν = Kinematic Viscosity	= 0.920×10^{-5} ft ² /sec

Table 2
Mean Flow Variables

y/D	\bar{U}	$\overline{u'u'}$	$\overline{v'v'}$	$-\overline{u'v'}$
0.775	1.460	0.00578	0.00274	0.00106
0.452	1.350	0.00922	0.00417	0.00251
0.290	1.260	0.01232	0.00578	0.00328
0.208	1.190	0.01513	0.00711	0.00395
0.129	1.090	0.01796	0.00805	0.00404
0.071	0.960	0.02560	0.01124	0.00409

Initial input for the model consisted of both the mean flow variables for each grid point and the boundary conditions. The mean flow values of \bar{U} , $\overline{u'u'}$, $\overline{v'v'}$, and $-\overline{u'v'}$ were obtained from curves plotted through the experimental values shown on Table 2. The appropriate grid point value was then merely picked from the respective profile plot. Unfortunately, values of $\overline{w'w'}$ were not taken; therefore, initial estimates were made by assuming a relationship between $\overline{u'u'}$, $\overline{v'v'}$, and $\overline{w'w'}$ similar to that shown by Comte-Bellot's data (1965) as shown by Launder, Reece, and Rodi (1975). This relationship places the $\overline{w'w'}$ value at approximately one-third of the difference $(\overline{u'u'} - \overline{v'v'})$ above the value of $\overline{v'v'}$. Starting values for $\overline{w'w'}$ were then estimated from the plots of the experimental data. The initial mean flow velocity profile and water-level elevation for the entire

flow section was placed at a) $\bar{V}(y) = 0$ (vertical velocity), b) $\bar{U}(y)$ as plotted from Table 2, and c) $D(x) = 0.1013$ ft (depth of flow). The additional parameter of bottom slope had to be deduced from the mean flow parameters (Table 1) since it was not provided in the initial data. The reason for this is that the slope of the energy grade line (Table 1) is greater than the friction slope calculated from the Chezy discharge coefficient, indicating gradually varying flow as opposed to steady uniform flow. A slightly lower Chezy discharge coefficient of 19.5 was therefore selected which yielded an intermediate slope value of 0.00150. This value is more realistic of the slope required to produce a normal depth of 0.1013 ft, although it is not exact. This is shown from the model output which does indicate a slight gradually varied flow condition. This could be rectified by trial-and-error determinations; however, it was not felt necessary since the experimental data also indicated gradually varied flow.

Additional parameters necessary for model input are those constants used in the appropriate closure schemes mentioned in Section 5. The resulting values were not computer optimized, but were determined by consideration of both published values and model output. Table 3 shows a comparison of the selected values with those published values.

Table 3
Closure Approximation Coefficients

Symbol	Present Model	Hanjalic & Launder	Launder, Reece, & Rodi
C_1	2.00	2.80	1.50
C_2	0.50	0.45	0.40
C_s	0.10	0.08	0.11
C_3	2.50	--	--

Computational variables for this selected flow regime are as follows:

$$\Delta x = 2.0 \text{ ft}$$

$$\Delta y = 1/10 \text{ depth of flow}$$

$$\Delta t = 0.50 \text{ sec}$$

$$\Delta \tau = 0.01 \text{ sec}$$

These values were based on computational speed and stability criteria and were not computer optimized. All computations were performed on a Texas Instruments Model 980-B mini-computer with 64 K bytes of memory. Peripheral equipment included a 2.5 megabyte moving head disk, dual cassette transports, and an electrostatic printer/plotter. The vertical mean velocity and Reynolds stress distributions resulting from the model are shown in Figures 1 and 2. The mean velocity profile shows an inflection at the surface which is probably due to the gradually varied flow condition

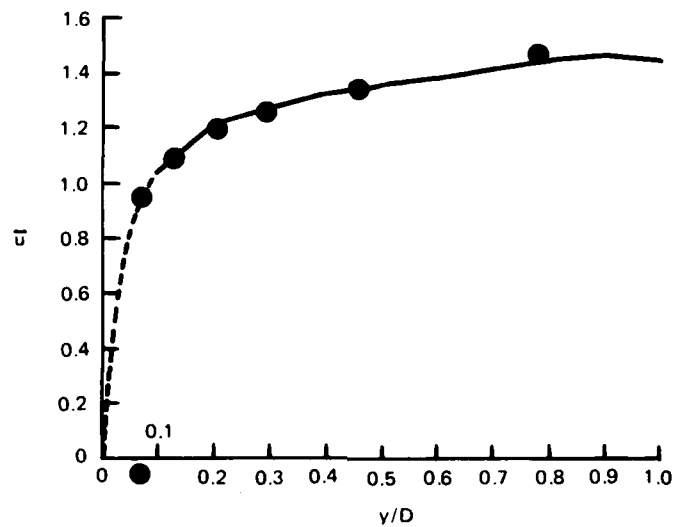


Figure 1. Mean velocity profile

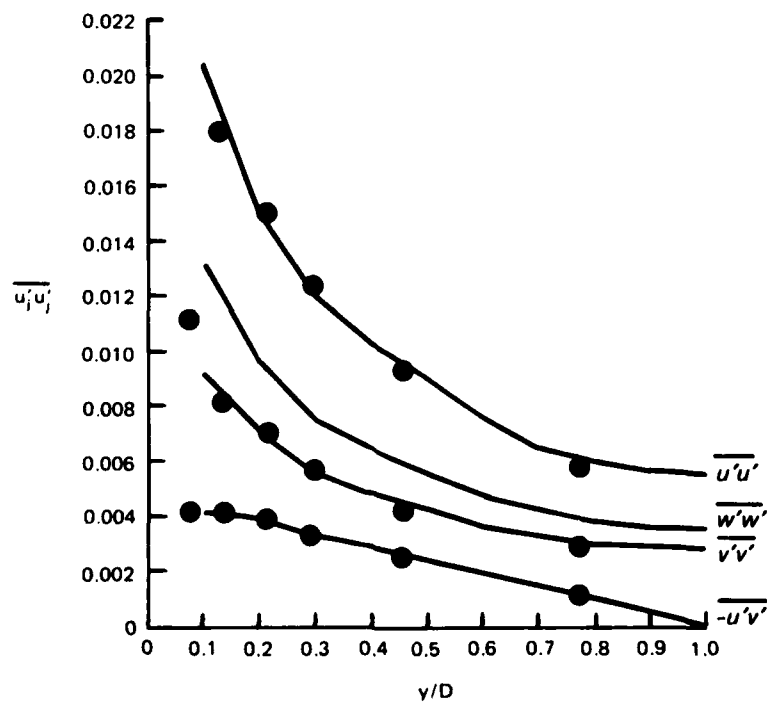


Figure 2. Reynolds stresses

resulting from the channel bed slope selection. With this exception, the agreement is acceptable. The Reynolds stress profiles are in excellent agreement with the experimental results. Values generated for $\overline{w'w'}$ are included although experimental results were not provided for comparison. Figures 3-6 represent plots showing the computed distribution of production, dissipation, pressure redistribution, and diffusion for each of the Reynolds stress equations. Figure 7 represents a consolidation of those individual terms for just the normal Reynolds stress equations (Figures 3-5). No direct comparisons can be made to experimental data since these terms are not directly measurable.

Inspection of the individual terms in the Reynolds stress transport equations (Figures 3-6) show some interesting trends which can not be deduced from other model results. For example, the predominant terms in the equations for $v'v'$ and $w'w'$ are dissipation and pressure redistribution, while production, pressure redistribution, and dissipation are significant in the $u'u'$ equation. This would appear reasonable since the $v'v'$ and $w'w'$ equations contain no production terms; therefore any turbulent energy entering from redistribution must be balanced by dissipation. Examination of the $u'u'$ equation shows that the production term is balanced by a combination of pressure redistribution and dissipation. The $u'v'$ equation, which also contains a production type term, shows that dissipation is relatively insignificant and that production is balanced by pressure

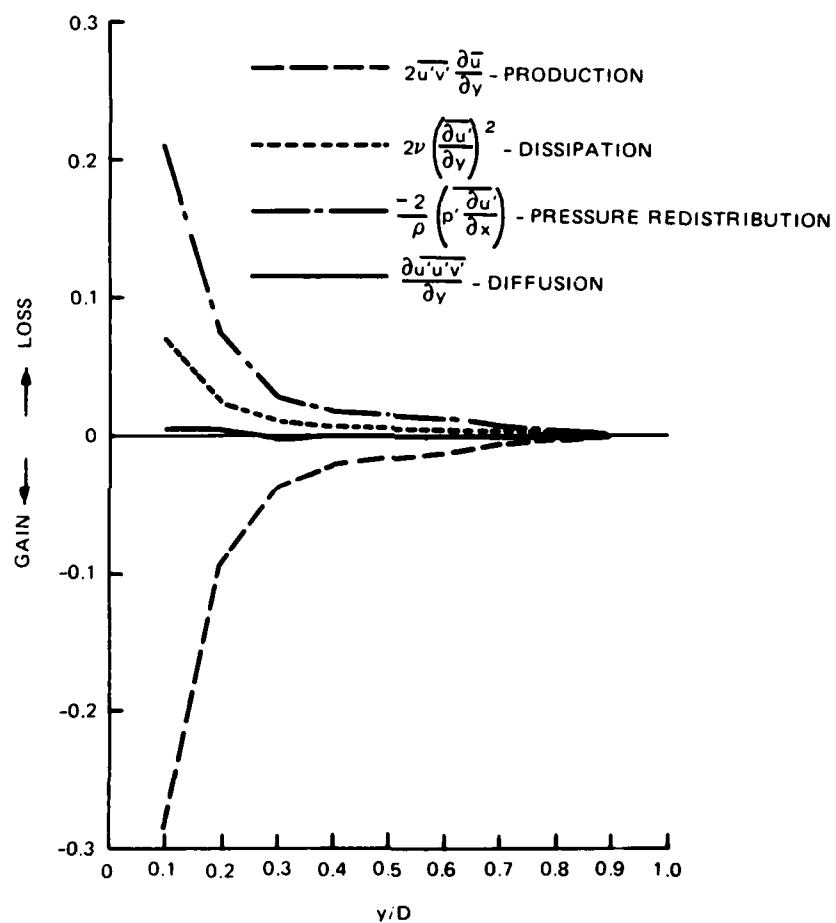


Figure 3. $\overline{u'u'}$ energy balance

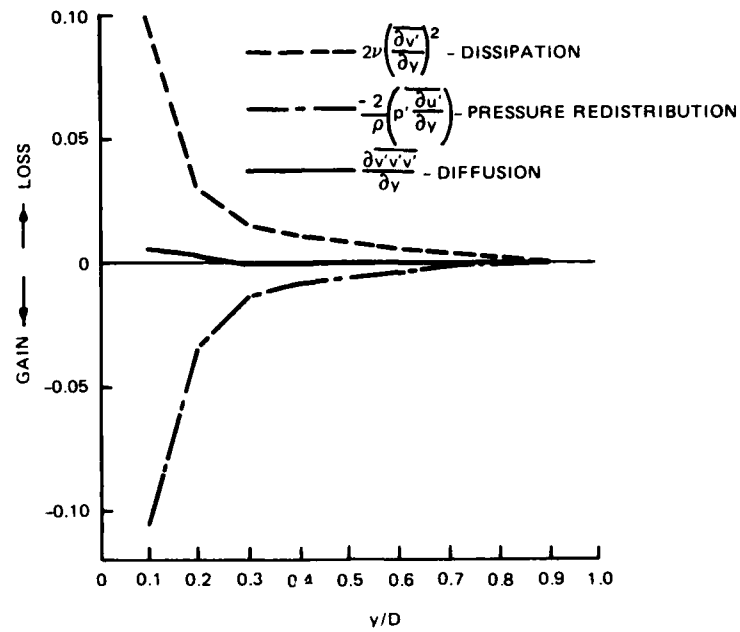


Figure 4. $\overline{v'v'}$ energy balance

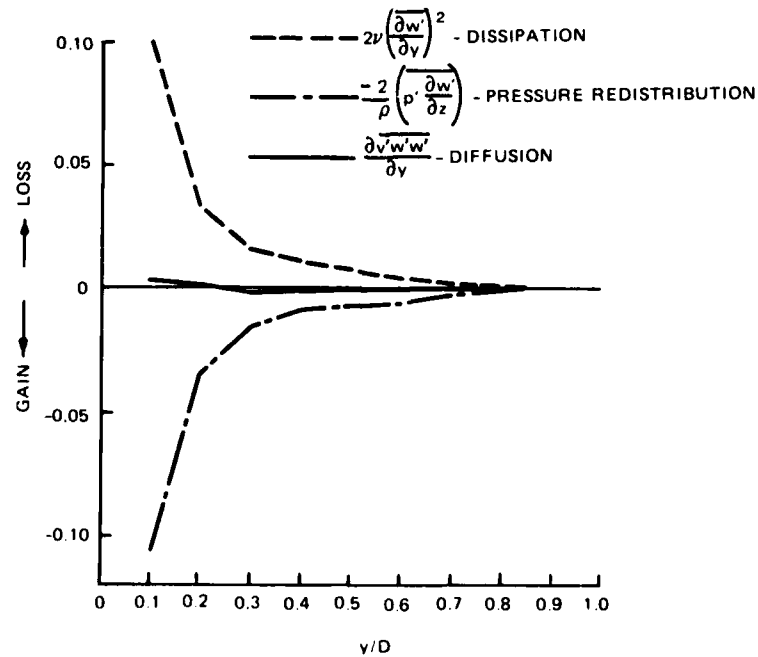


Figure 5. $\overline{w'w'}$ energy balance

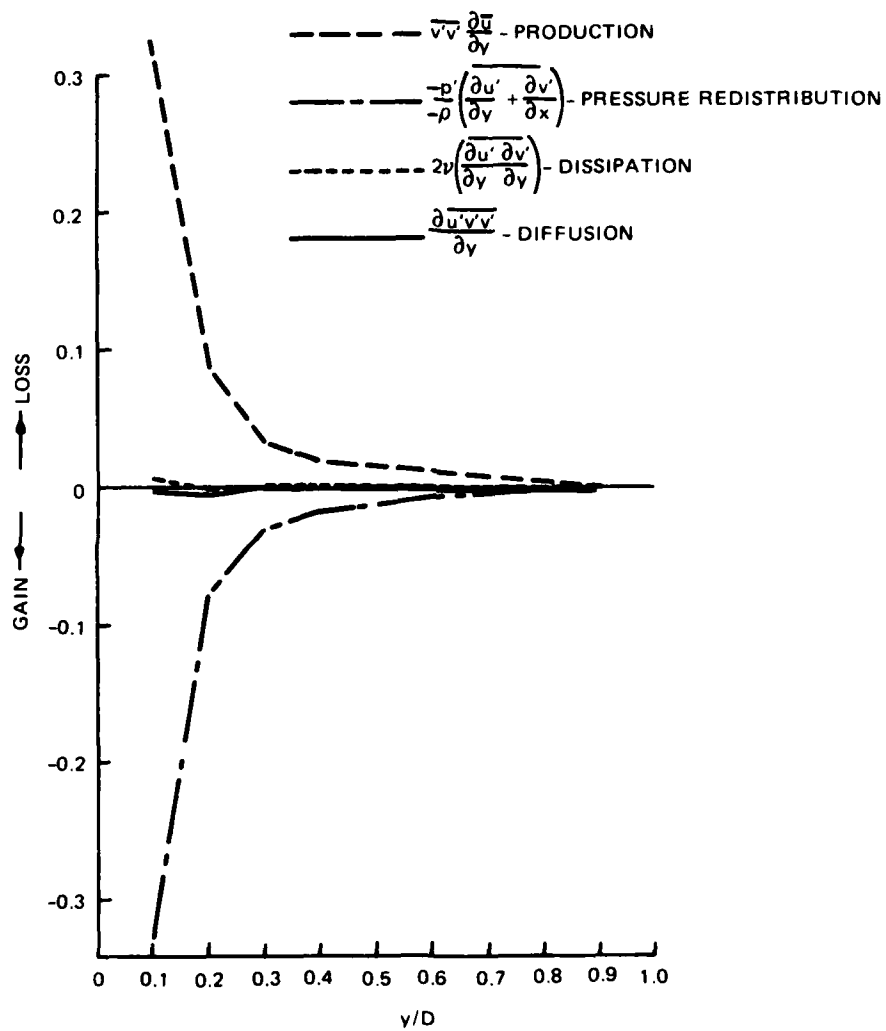


Figure 6. $\overline{u'v'}$ energy balance

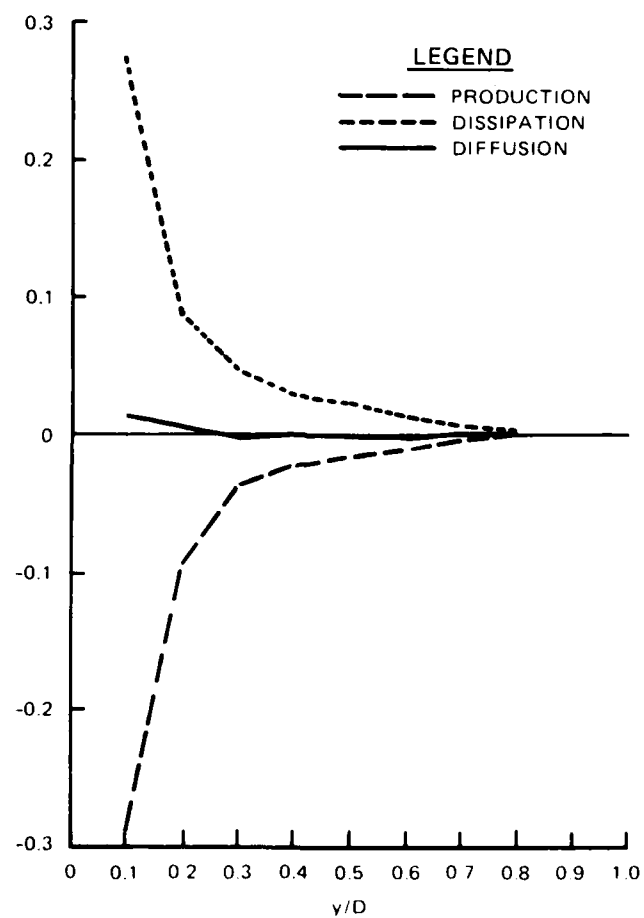


Figure 7. $\overline{u'_k u'_k}$ energy balance

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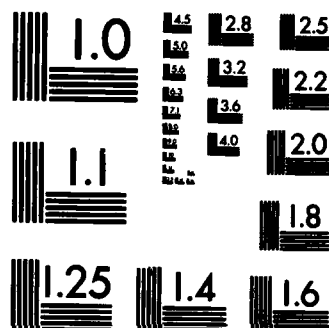
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redistribution. In all cases, the diffusion term is small as was shown in the perturbation expansions of Section 4. The usual assumption of isotropic dissipation can now be evaluated by comparing the appropriate terms in the four plots. The condition of negligible dissipation in the $u'v'$ equation can be seen; however, dissipation in the $u'u'$ equation is significantly lower than the dissipation in the equations for $v'v'$ and $w'w'$. This would lead to the conclusion, based on this formulation, that turbulent dissipation is not isotropic and that the assumption of isotropy could lead to significant errors in the overall model results.

The above conclusions have pointed out that the behavior of turbulent fluctuations in a shallow water wave climate is almost entirely a function of the processes of production, pressure redistribution, and dissipation. Of these three quantities, only production can be expressed in terms of known quantities. The remaining terms must be approximated through an appropriate closure scheme. Sources of error introduced through inaccurate approximations are difficult to detect due to the high degree of interdependency of the Reynolds transport equations. Nevertheless, an assessment of the closure schemes used in this model will be made based on the final output of the model.

1. Diffusion. Although it was shown that the diffusion terms were of a low order of significance, they were included to render the set of equations parabolic. The

inclusion of these terms, especially near the wall, are necessary; however, it is felt that a simpler formulation could easily be used. The computational effort required for this approximation is much greater than that required for the other closure schemes. This is due not only to the physical length of the approximation but also to the fact that the coefficient k/ϵ may approach infinity near the surface. This possibility must be accounted for. Due to the relative magnitude of the term, and the fact that its accuracy has been questioned (Launder, Reece, and Rodi, 1975), the above observations lead to the conclusion that a simpler approximation should be used.

2. Pressure Redistribution. The pressure redistribution terms are probably the most important terms in the set of equations since they are significant in all equations. They are also the most difficult to approximate since they deal with a pressure and velocity fluctuation correlation term which cannot even be measured in the laboratory. The closure scheme selected (Launder, Reece, and Rodi, 1975) appears to give very reasonable results. This conclusion is based on both the magnitude of the approximation and the response of the approximation to changes in other flow variables. The use of the simpler near-wall correction for the $u'v'$ equation also was shown to be quite effective for that region. In view of the above, it

is concluded that the pressure redistribution closure scheme is a very effective one. This conclusion was also reached by Laufer, Reece, and Rodi (1975).

3. Dissipation. The dissipation approximations presented in this paper hopefully represent an improvement in turbulence modeling in that the assumption of isotropic dissipation is not made. The derivations for the approximations are exact and no major assumptions or simplifications were made. The resulting approximations (equations 6.15-6.18) can be simplified further since the first term is relatively insignificant compared with the term involving the two-point velocity correlation coefficient derivatives. The normal dissipation approximations then reduce to

$$\overline{\left(\frac{\partial u_i'}{\partial y}\right)^2} = - f_i'' \overline{u_i' u_i'} \quad 8.1$$

where summation of indices is not intended. Although these coefficient derivatives (f_i'') were initially approximate from the experimental data, for this model they can be measured in the laboratory. This has been performed in a 5-inch wind tunnel by Laufer (1951) using the following relationship, identical with equation 8.1, in which he defines a microscale of turbulence for the $u'u'$ velocity fluctuation as

$$\frac{\overline{u'u'}}{\lambda_y^2} \equiv \frac{1}{2} \left(\frac{\partial \overline{u'}}{\partial y} \right)^2 = - \frac{1}{2} \left(\frac{\partial^2 R_y^{u'}}{\partial y^2} \right) \overline{u'u'} \quad 8.2$$

where $R_y^{u'}$ is the appropriate correlation coefficient for $\overline{u'u'}$. This is a consequence of the assumption that

$$R_{u'} = e^{-\frac{y^2}{\lambda^2}} \quad 8.3$$

Using the same assumption, equivalent λ values can be plotted for the two-point velocity correlation coefficients used in this model. Figure 8 shows a comparison of the model values corresponding to the three Reynolds stress values in addition to the data corresponding to $\overline{u'u'}$, as measured by Laufer. Although absolute values are not comparable, because of experimental differences, the plot does demonstrate that the derived model values are of the same order of magnitude. This is also confirmed by similar plots of λ values for grid turbulence reported by Batchelor and Townsend (1948) as shown in Cebeci and Smith (1974, p. 19). This data shows maximum λ values of approximately 0.44. Although the concept of using velocity correlation coefficients for dissipation approximations is not new, it has not been applied to numerical turbulence modeling. In view of Figure 2, it is apparent that it can be successfully used for this purpose and does represent an improvement over isotropic dissipation models.

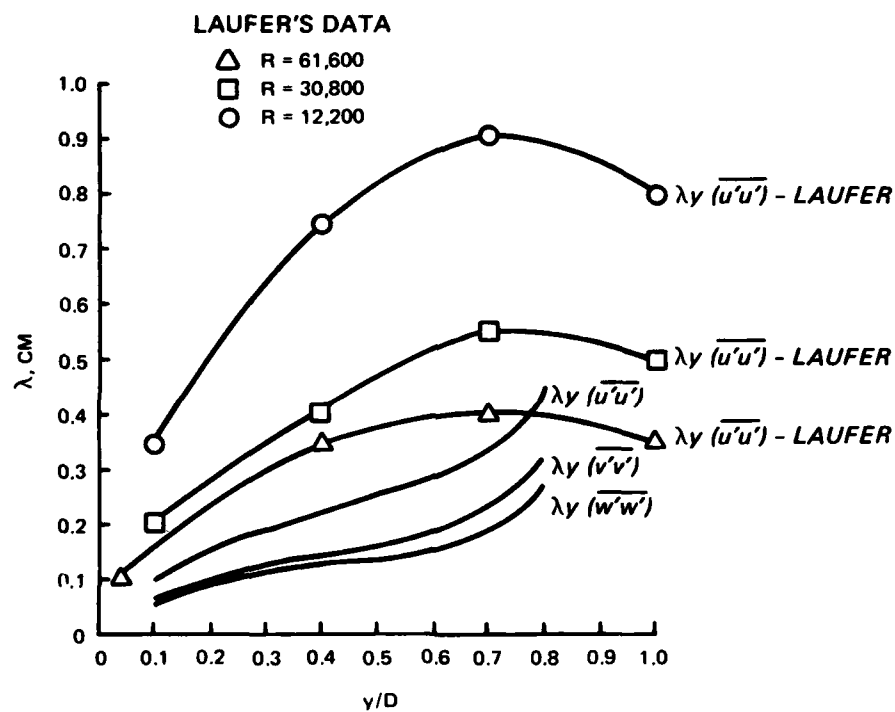


Figure 8. Turbulence microscale values

The overall conclusion regarding the verification of the model is that the derived governing equations will acceptably reproduce the turbulent flow regime for a shallow-water wave situation. This has been shown through application of the model to an experimentally derived set of data in which very good reproduction of the laboratory data was achieved by the model. The numerical methods employed in this model and the efficiency of these calculations were not stressed since the goal of this project was to demonstrate the applicability of the perturbation approach for a turbulent flow condition with practical applications. This goal has been realized.

SECTION 9

CONCLUSIONS

A two-dimensional, second-order closure, Reynolds stress turbulence model has been developed for the investigation of tidally induced flow regimes. The governing equations for this model have been developed through the application of a formal perturbation expansion of the scaled and nondimensionalized turbulent flow equations. Existing closure schemes were used for the pressure redistribution and diffusion terms in the Reynolds stress transport equations; however, a non-isotropic closure formulation was developed for the approximation of the dissipation terms. This approach represents an improvement over existing turbulence models, which currently use an isotropic dissipation closure scheme. This feature was developed in the belief that turbulent shear flows are not isotropic and that to consider them so would severely limit the versatility of the model. The concluding remark by Launder, Reece, and Rodi (1975) that "...an assessment of the validity of the assumption of local isotropy as the most urgent research task in extending further the range of applicability of the present model," would seem to reinforce this

viewpoint. This versatility is essential if the model is to be applicable to complex flows.

A finite difference approximation numerical model was written to solve the seven governing equations for the mean velocities \bar{u} and \bar{v} , the Reynolds stresses $\overline{u_i' u_j'}$, and the water-surface elevation $\bar{\delta}$. This model was then verified to an existing set of laboratory data for the case of turbulent steady flow in a flume. The predicted mean and turbulent flow quantities agreed well with the measured experimental values. Slight disagreements may be attributed to the following: 1) the experimental data were sparse with only six measurements taken in the vertical, and 2) steady uniform flow was apparently not reached in the 10-meter-long flume. This resulted in steady, gradually varied flow in which the velocity just below the surface may be greater than that at the surface. This was the result shown by the numerical model. In view of the above, the accuracy of the data, and the numerical methods used in the model, it is concluded that the derived set of governing equations and closure equations do acceptably approximate a steady turbulent flow regime.

The present investigation has shown the following:

1. The turbulent quantity term in the x-direction momentum equation (EQ. 6.4) is significant in magnitude with respect to the mean flow quantities. Improper modeling of this term will lead to inaccuracies in the calculations for mean flow quantities.

2. Acceleration terms and derivatives with respect to the direction of flow in the Reynolds stress transport equations are small in comparison to other quantities. This shows that changes of turbulent quantities with respect to the vertical direction are more significant than changes with respect to the direction of flow. This observation is a result of the fact that tidally induced accelerations in the flow field are small. This result would not be valid for steep waves causing a rapidly varying flow condition.
3. The nonisotropic dissipation closure approximations developed for this project are superior, in theory, to the usual assumption of isotropy. The approximation does produce acceptable results and can be seen to be nonisotropic in Figures 3-5. Extensions of the model to nonsteady-state conditions should demonstrate the superiority of this closure approach. The potential importance of this term can also be seen by the fact that it contains the fluid density in the kinematic viscosity. This could be an important consideration for future extensions of the model for density stratification and/or sediment transport studies.
4. The pressure redistribution closure approximation appears to yield satisfactory results. The diffusion closure approximation, however, has the possibility of becoming unstable in areas of low turbulence. For this

reason, slight alterations to the present formulation will probably have to be made if the model is to be tested for nonsteady-state flows.

The present model was formulated to provide a basic understanding of the hydrodynamics of a turbulent, tidally induced flow. Without this insight, it would be difficult to conduct a meaningful investigation of other flow-related phenomena such as density stratification or the nonlinear propagation of tides. The model developed for this project has been shown to adequately represent the various mechanisms of steady turbulent flow. The extension to nonsteady flow should not be too difficult since the basic formulation was for that specific application. Following verification to nonsteady conditions, the model can be used in conjunction with additional equations for the detailed study of other flow-related problems. An additional benefit of the model is that the concepts developed in the formulation of the model can be used to improve upon existing hydrodynamic models. In either application, the model will have produced an increased understanding of turbulent flow which will ultimately aid in the development of improved predictive models for estuarine flow.

REFERENCES

1. Launder, B.E., and Spalding, D.B. 1972. Lecture in Mathematical Models of Turbulence, Academic Press.
2. Rodi, W. 1980 (June), "Turbulence Models and Their Application in Hydraulics," International Association for Hydraulic Research, State-of-the-Art Paper.
3. Smith, T.J., and Takhar, H.S. 1979. "On the Calculation of the Width Averaged Flow Due to Long Waves in an Open Channel," Journal of Hydraulic Research, No. 4.
4. Hanjalic, K., and Launder, B.E. 1972. "A Reynolds Stress Model of Turbulence and its Application to Thin Shear Flows," Journal of Fluid Mechanics, Vol. 52, Part 4.
5. Launder, B.E., Reece, G.T., and Rodi, W. 1975. "Progress in the Development of a Reynolds-Stress Turbulence Closure," Journal of Fluid Mechanics, Vol. 68, Part 3.
6. Legner, H.H., and Finson, M.L. 1980. "On the Stability of Fine-Scaled Turbulent Free Shear Flows," Journal of Fluid Mechanics, Vol. 100, Part 2.
7. Airy, G.B. 1845. "Tides and Waves," Encyclopedia Metropolitana, Vol. 5, London.
8. Boussinesq, J. 1871. "Theorie De L'intumescence Liquide Appelee onde Solitaire ou de Translation se Propageant Dans un Canal Rectangulaire," C.R. Acad. Sci., Paris, 72.
9. Rayleigh, Lord. 1876. "On Waves," Philosophical Magazine, Edinburg, Series 5, Vol. 1.
10. Friedrichs, K.O. 1948. "On the Derivation of the Shallow Water Theory," Communications on Pure and Applied Mathematics, Vol. 1.
11. Keller, J.B. 1948. "The Solitary Wave and Periodic Waves in Shallow Water," Communications on Pure and Applied Mathematics, Vol. 1.

12. Wehausen, J.V., and Laitone, E.V. 1960. "Surface Waves," Encyclopedia of Physics, Volume IX, Fluid Dynamics III.
13. Ursell, F. 1953. "The Long Wave Paradox in the Theory of Gravity Waves," Proceedings of the Cambridge Philosophical Society, Vol. 49.
14. Prandtl, L. 1925. "Uber Die Ausgebildete Turbulenz," Z. Angew. Math. u. Mech., 5, p. 136.
15. Prandtl, L. 1945. "Uber Ein Neues Formelsystem Fur Die Ausgebildete Turbulenz," Nachr. Akad. Wiss., Gottingen, Math-Phys, p. 6.
16. Kolmogorov, A.N. 1942. "Equations of Turbulent Motion of an Incompressible Fluid," Izv. Akad. Navk. SSR Seria Fizicheskaya VI, No. 1-2. (English Translation: Imperial College, Mech. Eng. Dept. Rept. On/6, 1968.)
17. Townsend, A.A. 1956. The Structure of Turbulent Shear Flow, Cambridge, p. 234.
18. Hinze, J.O. 1959a. Turbulence, McGraw Hill, New York.
19. Schlitchting, H. 1960. Boundary Layer Theory, McGraw Hill, New York.
20. Batchelor, G.K. 1956. The Theory of Homogeneous Turbulence, Cambridge, p. 96.
21. Taylor, A.E. 1955. Advanced Calculus, Blaisdell Publishing Co., Waltham, Mass.
22. Chou, P.Y. 1945. "On Velocity Correlations and the Solutions of the Equations of Turbulent Fluctuation," Quart. Appl. Math., 3, 38.
23. Rotta, J.C. 1951. "Statistische Theorie Nichthomogener Turbulenz," Z. Phys., 129, 547.
24. Millionshtchikov, M.D. 1941. "On the Theory of Homogeneous Isotropic Turbulence," C.R. Acad. Sci. S.S.S.R., 32, p. 615-619.
25. Reichardt, H. 1938. "Messungen Turbulenter Schwan- kungen," Naturwissenschaften, p. 404.
26. Klebanoff, P.S. 1955. "Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient," NACA Rep. No. 1247.
27. Laufer, J. 1954. National Advisory Committee for Aeronautics, NACA Tech. Repts. 1174.

28. Hornbeck, R.W. 1975. Numerical Methods, Quantum Publishers, Inc., New York.
29. McQuivey, R.S. 1973. "Summary of Turbulence Data from Rivers, Conveyance Channels, and Laboratory Flumes," Geological Survey Professional Paper 802-B.
30. Laufer, J. 1951. "Investigation of Turbulent Flow in a Two-Dimensional Channel," National Advisory Committee for Aeronautics, Report 1053.
31. Batchelor, G.K., and Townsend, A.A. 1948. "Decay of Turbulence in the Final Period," Proc. Roy. Soc. A., 194A, 527.
32. Cebeci, T., and Smith, A.M.O. 1974. Analysis of Turbulent Boundary Layers, Academic Press, New York, San Francisco, London.
33. Noat, D., and Rodi, W. 1982. "Calculation of Secondary Currents in Channel Flow," Journal of the Hydraulics Division, A.S.C.E., No. HY8, August.
34. Lamb, H. 1932. Hydrodynamics, Dover Publications, 6th Edition.
35. Madsen, O.S., and Mei, C.C. 1969. "Dispersive Long Waves of Finite Amplitude over an Uneven Bottom," Mit Department of Civil Engineering, Report No. 117, November.
36. Daily, J.W., and Harleman, D.R.F. 1966. Fluid Dynamics, Addison-Wesley Publishing Co., Inc. Reading, Mass.
37. Stoker, J.J. 1957. Water Waves, Interscience Publishers, Inc., New York. p. 19.
38. Ippen, A.T. (Editor) 1966. Estuary and Coastline Hydrodynamics, McGraw Hill Book Co., Ch. 2 (Dean, R.G. and Eagleson, P.S.), p. 97.
39. Comte-Bellot, G. 1965. "Ecoulement Turbulent Entre Deux Parois Paralleles," Publ. Sci. Tech. Ministere De L'Air, No. 419.

APPENDIX A

THE REYNOLDS EQUATIONS AND THE CONTINUITY EQUATION

The equations governing the flow distribution of a fluid are based on the principle of conservation of mass and conservation of linear momentum. The two-dimensional equations for an incompressible, Newtonian fluid can be written as follows:

Conservation of momentum: Navier-Stokes equations

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \quad A.1$$

$$\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = - \frac{1}{\rho} \frac{\partial P}{\partial y} - g + \nu \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right) \quad A.2$$

Conservation of mass: continuity equation

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \quad A.3$$

where U and V represent the instantaneous horizontal and vertical velocities, P represents the instantaneous pressure, g is the negative body force due to gravity, and ρ and ν represent the fluid density and kinematic viscosity. The

above equations correctly describe turbulent motion; however, the time and length scales necessary to do so are beyond the practical capabilities of present computers. The accepted approach to describing the turbulent motion was first proposed by Osborn Reynolds. The suggested approach was that each flow variable should be represented as the sum of an average quantity plus a fluctuating quantity. This relationship can be expressed as follows

$$U = \bar{U} + u'$$

$$V = \bar{V} + v' \quad \text{A.4}$$

$$P = \bar{P} + p'$$

where the average quantities are defined as

$$\bar{U} = \frac{1}{T} \int_0^T U(t+\tau) d\tau, \quad \bar{V} = \frac{1}{T} \int_0^T V(t+\tau) d\tau, \quad \bar{P} = \frac{1}{T} \int_0^T P(t+\tau) d\tau$$

The magnitude of T is further defined as large in comparison to the time scales of turbulent motion. Substitution of equations A.4 into equations A.1, A.2, and A.3, and averaging the resulting quantities with respect to time yields the following equations:

$$\frac{\partial \bar{U}}{\partial t} + \bar{U} \frac{\partial \bar{U}}{\partial x} + \bar{V} \frac{\partial \bar{U}}{\partial y} = - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x} + \nu \left(\frac{\partial^2 \bar{U}}{\partial x^2} + \frac{\partial^2 \bar{U}}{\partial y^2} \right) - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y}$$

A.5

$$\frac{\partial \bar{V}}{\partial t} + \bar{U} \frac{\partial \bar{V}}{\partial x} + \bar{V} \frac{\partial \bar{V}}{\partial y} = - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial y} - g + \nu \left(\frac{\partial^2 \bar{V}}{\partial x^2} + \frac{\partial^2 \bar{V}}{\partial y^2} \right) - \frac{\partial \overline{u'v'}}{\partial x} - \frac{\partial \overline{v'v'}}{\partial y}$$

A.6

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0$$

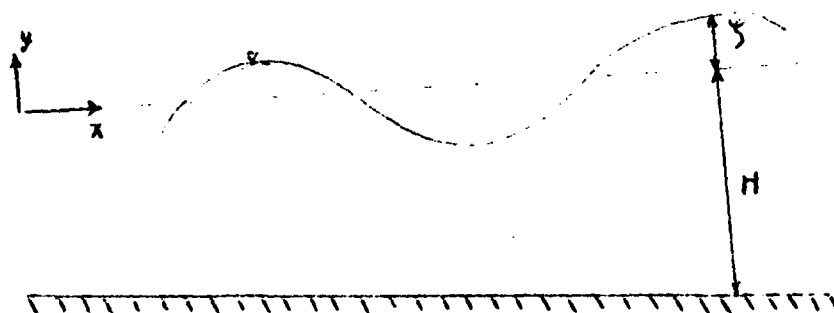
A.7

Equations A.5 and A.6 represent the well-known Reynolds equations and equation A.7 is the continuity equation. The fluctuating quantities are referred to as the Reynolds stresses. The goal of a turbulent model is to provide a solution to these equations for the mean flow quantities.

APPENDIX B

BOUNDARY CONDITIONS

Boundary conditions for the free surface and bottom of the flow region are specified based on the dimensions shown below (for two-dimensional flow)



where the free surface is defined as a wave superimposed on a mean water level. Boundary conditions are developed by defining a variable $F(x,y,t)$ for both the upper and lower surface which has the following property

$$\frac{dF}{dt} \equiv 0 \quad \text{on the boundary}$$

The surface variable can be defined as

$$F_s = y - \zeta(x,t)$$

The boundary equation can then be written as follows

$$\frac{dF_s}{dt} = \frac{\partial F_s}{\partial t} + U_s \frac{\partial F_s}{\partial x} + V_s \frac{\partial F_s}{\partial y} = 0$$

or

$$\frac{\partial \zeta}{\partial t} + U_s \frac{\partial \zeta}{\partial x} + V_s = 0 \quad \text{B.1}$$

where U and V represent velocities in the x and y directions and the subscript denotes the surface. The bottom variable can be defined as shown below

$$F_B = y + H(x)$$

The resulting equation can be written as

$$U_B \frac{\partial H}{\partial x} + V_B = 0 \quad \text{B.2}$$

A turbulent expansion, similar to that shown in Appendix A, yields the following equations

$$\frac{\partial \bar{\zeta}}{\partial t} + \bar{U}_s \frac{\partial \bar{\zeta}}{\partial x} + \overline{u'_s \frac{\partial \zeta'}{\partial x}} - \bar{V}_s = 0 \quad \text{B.3}$$

and

$$\bar{U}_B \frac{\partial \bar{H}}{\partial x} + \bar{V}_B = 0 \quad \text{B.4}$$

These two relationships define the flow boundaries for turbulent free surface flow in two dimensions.

APPENDIX C THE REYNOLDS STRESS TRANSPORT EQUATIONS

The transport equations for the Reynolds stresses $(\overline{u_i' u_j'})$ can be derived from the Navier-Stokes equations shown in equations A.1 and A.2. This can be accomplished by first writing the equation in indicial notation as follows:

$$\frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} = - \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} + F_i + \nu \frac{\partial^2 U_i}{\partial x_k \partial x_k} \quad \text{C.1}$$

where F_i represents the body force ($F_x = 0$, $F_y = -g$). As was shown in Appendix A, each variable is expressed as the sum of an average quantity plus a fluctuating quantity as shown below

$$\begin{aligned} U_i &= \bar{U}_i + u_i' \\ P &= \bar{P} + p' \end{aligned} \quad \text{C.2}$$

Substitution of the expansions of equations C.2 into equation C.1 gives the following

$$\begin{aligned}
& \frac{\partial \bar{U}_i}{\partial t} + \frac{\partial u'_i}{\partial t} + \bar{U}_k \frac{\partial \bar{U}_i}{\partial x_k} + u'_k \frac{\partial \bar{U}_i}{\partial x_k} + \bar{U}_k \frac{\partial u'_i}{\partial x_k} + u'_k \frac{\partial u'_i}{\partial x_k} \\
& = \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x_i} - \frac{1}{\rho} \frac{\partial p'}{\partial x_i} + F_i + \nu \frac{\partial^2 \bar{U}_i}{\partial x_k \partial x_k} + \nu \frac{\partial^2 u'_i}{\partial x_k \partial x_k}
\end{aligned} \tag{C.3}$$

The last term on the left side of this equation can be written in the following manner

$$u'_k \frac{\partial u'_i}{\partial x_k} = \frac{\partial u'_i u'_k}{\partial x_k} - u'_i \frac{\partial u'_k}{\partial x_k} = \frac{\partial u'_i u'_k}{\partial x_k}$$

by use of the continuity equation for the velocity fluctuations

$$\frac{\partial u'_k}{\partial x_k} = 0$$

Equation C.3 can now be written in the following form

$$\begin{aligned}
& \frac{\partial \bar{U}_i}{\partial t} + \frac{\partial u'_i}{\partial t} + \bar{U}_k \frac{\partial \bar{U}_i}{\partial x_k} + u'_k \frac{\partial \bar{U}_i}{\partial x_k} + \bar{U}_k \frac{\partial u'_i}{\partial x_k} + \frac{\partial u'_i u'_k}{\partial x_k} \\
& = - \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x_i} - \frac{1}{\rho} \frac{\partial p'}{\partial x_i} + F_i + \nu \frac{\partial^2 \bar{U}_i}{\partial x_k \partial x_k} + \nu \frac{\partial^2 u'_i}{\partial x_k \partial x_k}
\end{aligned} \tag{C.4}$$

If the averaging process described in Appendix A is applied to this relationship, the following mean value equation results:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_k \frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial \overline{u'_i u'_k}}{\partial x_k} = - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \bar{F}_i + \nu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_k} \quad C.5$$

Substitution of equation C.5 from C.4 results in an equation for the transport of the single velocity fluctuation u'_i .

This relationship becomes:

$$\frac{\partial u'_i}{\partial t} + u'_k \frac{\partial \bar{u}_i}{\partial x_k} + \frac{\partial}{\partial x_k} (u'_i u'_k - \overline{u'_i u'_k}) = - \frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u'_i}{\partial x_k \partial x_k} \quad C.6$$

This can also be written for the j index as:

$$\frac{\partial u'_j}{\partial t} + u'_k \frac{\partial \bar{u}_j}{\partial x_k} + \frac{\partial}{\partial x_k} (u'_j u'_k - \overline{u'_j u'_k}) = - \frac{1}{\rho} \frac{\partial p'}{\partial x_j} + \nu \frac{\partial^2 u'_j}{\partial x_k \partial x_k} \quad C.7$$

multiplication of equation C.6 by u'_j , multiplication of equation C.7 by u'_i , and addition of the two results become:

$$\begin{aligned} & u'_j \frac{\partial u'_i}{\partial t} + u'_i \frac{\partial u'_j}{\partial t} + u'_j u'_k \frac{\partial \bar{u}_i}{\partial x_k} + u'_i u'_k \frac{\partial \bar{u}_j}{\partial x_k} + u'_j \bar{u}_k \frac{\partial u'_i}{\partial x_k} + u'_i \bar{u}_k \frac{\partial u'_j}{\partial x_k} \\ & + u'_j \frac{\partial}{\partial x_k} (u'_i u'_k - \overline{u'_i u'_k}) + u'_i \frac{\partial}{\partial x_k} (u'_j u'_k - \overline{u'_j u'_k}) \\ & = - \frac{1}{\rho} u'_j \frac{\partial p'}{\partial x_i} - \frac{1}{\rho} u'_i \frac{\partial p'}{\partial x_j} + \nu u'_j \frac{\partial^2 u'_i}{\partial x_k \partial x_k} + \nu u'_i \frac{\partial^2 u'_j}{\partial x_k \partial x_k} \end{aligned} \quad C.8$$

The following identities can be seen:

$$u'_j \frac{\partial u'_i}{\partial t} + u'_i \frac{\partial u'_j}{\partial t} = \frac{\partial u'_i u'_j}{\partial t}$$

and

$$u_j' \bar{u}_k \frac{\partial u_i'}{\partial x_k} + u_i' \bar{u}_k \frac{\partial u_j'}{\partial x_k} = \bar{u}_k \frac{\partial u_i' u_j'}{\partial x_k}$$

Substitution of these relationships into equation C.8 and averaging the results yields the following:

$$\begin{aligned} & \frac{\partial \overline{u_i' u_j'}}{\partial t} + \overline{u_j' u_k'} \frac{\partial \bar{u}_i}{\partial x_k} + \overline{u_i' u_k'} \frac{\partial \bar{u}_j}{\partial x_k} + \bar{u}_k \frac{\partial \overline{u_i' u_j'}}{\partial x_k} + \overline{u_j' \frac{\partial}{\partial x_k} u_i' u_k'} \\ & + \overline{u_i' \frac{\partial u_j' u_k'}{\partial x_k}} = - \frac{1}{\rho} \overline{u_j' \frac{\partial p'}{\partial x_i}} - \frac{1}{\rho} \overline{u_i' \frac{\partial p'}{\partial x_j}} + \nu \overline{u_j' \frac{\partial^2 u_i'}{\partial x_k \partial x_k}} \\ & + \nu \overline{u_i' \frac{\partial^2 u_j'}{\partial x_k \partial x_k}} \end{aligned} \quad C.9$$

The following expansions can now be introduced:

$$\begin{aligned} \text{a)} \quad & \overline{u_j' \frac{\partial u_i' u_k'}{\partial x_k}} + \overline{u_i' \frac{\partial u_j' u_k'}{\partial x_k}} = \overline{u_j' \frac{\partial u_i' u_k'}{\partial x_k}} + \overline{u_i' u_k' \frac{\partial u_j'}{\partial x_k}} = \frac{\partial \overline{u_i' u_j' u_k'}}{\partial x_k} \\ \text{b)} \quad & \overline{u_j' \frac{\partial^2 u_i'}{\partial x_k \partial x_k}} + \overline{u_i' \frac{\partial^2 u_j'}{\partial x_k \partial x_k}} = \frac{\partial^2 \overline{u_i' u_j'}}{\partial x_k \partial x_k} - 2 \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} \\ \text{c)} \quad & \overline{u_j' \frac{\partial p'}{\partial x_i}} + \overline{u_i' \frac{\partial p'}{\partial x_j}} = \frac{\partial \overline{u_j' p'}}{\partial x_i} + \frac{\partial \overline{u_i' p'}}{\partial x_j} - \overline{p' \frac{\partial u_j'}{\partial x_i}} - \overline{p' \frac{\partial u_i'}{\partial x_j}} \end{aligned}$$

Substitution of these expansions into equation C.9 results in the following familiar form of the Reynolds stress transport equations:

$$\begin{aligned}
& \frac{\partial \overline{u_i' u_j'}}{\partial t} + \overline{u_j' u_k'} \frac{\partial \overline{u_i'}}{\partial x_k} + \overline{u_i' u_k'} \frac{\partial \overline{u_j'}}{\partial x_k} + \overline{u_k'} \frac{\partial \overline{u_i' u_j'}}{\partial x_k} \\
& \quad \text{I} \qquad \qquad \qquad \text{II} \qquad \qquad \qquad \text{I} \\
& = \frac{\overline{p'}}{\rho} \frac{\partial \overline{u_j'}}{\partial x_i} - \frac{\overline{p'}}{\rho} \frac{\partial \overline{u_i'}}{\partial x_j} - \frac{1}{\rho} \frac{\partial \overline{p' u_j'}}{\partial x_i} - \frac{1}{\rho} \frac{\partial \overline{p' u_i'}}{\partial x_j} \\
& \quad \qquad \qquad \text{III} \qquad \qquad \qquad \text{IV} \\
& + \nu \frac{\partial^2 \overline{u_i' u_j'}}{\partial x_k \partial x_k} - 2\nu \frac{\partial \overline{u_i'}}{\partial x_k} \frac{\partial \overline{u_j'}}{\partial x_k} - \frac{\partial \overline{u_i' u_j' u_k'}}{\partial x_k} \\
& \quad \qquad \qquad \text{V} \qquad \qquad \qquad \text{VI} \qquad \qquad \qquad \text{VII}
\end{aligned} \tag{C.10}$$

The usual interpretation of the various terms is presented below:

- I: These two terms represent the total rate of change of the turbulent fluctuation (d/dt). The first represents the time rate of change while the second represents the convection of the turbulent fluctuation.
- II: These terms, usually referred to as "production" terms, represent the transfer of turbulent energy in the system by providing a link between mean flow quantities and turbulent quantities (Reynolds stresses).
- III: Terms III are usually referred to as the "pressure redistribution" terms. This pressure rate-of-strain correlation has the effect of transferring energy in

the system to a more isotropic condition. When complete isotropy is achieved, it can be seen that this term vanishes due to the continuity equation (i.e., when $i = j$).

- IV: Terms IV are referred to as pressure-diffusion terms. They are generally much smaller in magnitude than the pressure redistribution terms and are therefore usually neglected.
- V: These terms represent diffusion of the Reynolds stresses through viscous action. They are also generally small in magnitude.
- VI: The terms labeled VI are referred to as the "dissipation" terms. They are responsible for dissipating turbulent energy through the viscous interactions of the rate of strain correlations.
- VII: The triple correlations of term VII represent the convection of the turbulent fluctuations by the turbulent fluctuations. This term will be referred to as the "diffusion" term.

Equations C.10 are the governing equations for the transport of the turbulent Reynolds stresses. The solution, or approximation, of these equations is the ultimate goal of a turbulence model.

APPENDIX D

THE POISSON EQUATION FOR PRESSURE FLUCTUATIONS

The divergence of the Navier-Stokes equations yields a Poisson equation for the pressure term for the case of incompressible flow. This relationship can be written for the two-dimensional case as follows:

$$-\frac{1}{\rho} \nabla^2 \frac{P}{\rho} = \frac{\partial^2 UU}{\partial x^2} + 2 \frac{\partial^2 UV}{\partial x \partial y} + \frac{\partial^2 VW}{\partial y^2} \quad D.1$$

As was shown in Appendix A, each variable can be expressed as the sum of an average quantity plus a turbulent fluctuating quantity, as shown below

$$U = \bar{U} + u'$$

$$V = \bar{V} + v'$$

$$P = \bar{P} + p'$$

Substitution of these expansions into equation D.1 yields the following:

$$\begin{aligned}
-\frac{1}{\rho} \nabla^2 \bar{p} - \frac{1}{\rho} \nabla^2 p' &= \frac{\partial^2 \overline{uu}}{\partial x^2} + 2 \frac{\partial^2 \overline{u}u'}{\partial y^2} + \frac{\partial^2 u'u'}{\partial x^2} + 2 \frac{\partial^2 \overline{uv}}{\partial x \partial y} \\
&+ 2 \frac{\partial^2 \overline{u}v'}{\partial x \partial y} + 2 \frac{\partial^2 u'v'}{\partial x \partial y} + \frac{\partial^2 \overline{vv}}{\partial y^2} + 2 \frac{\partial^2 \overline{v}v'}{\partial y^2} + \frac{\partial^2 v'v'}{\partial y^2}
\end{aligned} \tag{D.2}$$

If equation D.1 is averaged and then subtracted from equation D.2, the resulting equation is a Poisson equation for the pressure fluctuation. This relationship can be written as follows

$$\begin{aligned}
-\frac{1}{\rho} \nabla^2 p' &= \frac{\partial^2}{\partial x^2} (2\overline{u}u' + u'u') + 2 \frac{\partial^2}{\partial x \partial y} (\overline{u}v' + \overline{v}u' + u'v') \\
&+ \frac{\partial^2}{\partial y^2} (2\overline{v}v' + v'v')
\end{aligned} \tag{D.3}$$

Equation D.3 can be rearranged, by use of the continuity equation, and expressed in the following indicial notation:

$$-\frac{1}{\rho} \frac{\partial^2 p'}{\partial x_k \partial x_k} = \frac{\partial^2 u'_i u'_j}{\partial x_i \partial x_j} + 2 \frac{\partial \overline{u}_i}{\partial x_j} \frac{\partial u'_j}{\partial x_i} \tag{D.4}$$

The above equation is that used by Chou (1945) for the exact form of the pressure fluctuation - rate of strain correlation shown in equation 5.14. Although a closed-form solution is not possible, the use of equation D.4 has led to satisfactory approximations for the pressure fluctuation term in the Reynolds stress transport equations.

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